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**Curves and laminations  
on the five-times punctured sphere**

by

**Esmee Riet te Winkel**

**Thesis**

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# Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. I declare that the work of this thesis is my own, except where otherwise indicated in the text, or where the material is widely known. No part has been submitted by me for any other degree.

The illustrations were created using Tikz, Inkscape and Geogebra. The colours used correspond to lines 1 to 5 of the Paris Métro [Rat]. We were inspired by [DuG] to use this colour palette.

# Abstract

We study laminations on the five-times punctured sphere  $\Sigma_{0,5}$ . The discussion is divided into two parts. The results obtained in the two parts are not directly linked. In particular, each part can be read independently.

Firstly, we analyse the inclusion of the curve graph of  $\Sigma_{0,5}$  into  $\mathcal{PML}(\Sigma_{0,5})$ . We completely characterise the image of the induced subgraph on a pentagon and the associated decagon. This enables us to describe explicit loops in the curve graph whose images in  $\mathcal{PML}(\Sigma_{0,5})$  form a Hopf link and a trefoil knot.

Secondly, we investigate the topology of superconvergence  $\mathbf{T}_s$  on the set of boundary laminations  $\mathcal{BL}$ . Here a *boundary lamination* is a minimal lamination on  $\Sigma_{0,5}$  with more than one leaf. We show that  $(\mathcal{BL}, \mathbf{T}_s)$  is path connected. Gabai proved that the space of ending laminations is locally path connected [Gab1]. Using Gabai's theorem, we prove that  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected. Combined with work of Brock and Masur [BroM], this implies that the Gromov boundary of the pants graph of  $\Sigma_{0,5}$  is path connected and locally path connected.

# Abbreviations

## Standard notations

Symbol	Description
$\tilde{X}$	the universal cover of $X$
$\text{cl}(X)$	the closure of $X$
$\text{int}(X)$	the interior of $X$
$\text{fr}(X)$	the frontier of $X$ , that is, $\text{cl}(X) - \text{int}(X)$
$\text{Homeo}(X)$	the homeomorphism group of $X$
$\cong$	is homeomorphic to
$\mathbb{N}$	the natural numbers $1, 2, 3, \dots$
$\mathbb{Z}$	the integers
$\mathbb{C}$	the complex plane
$\mathbb{R}^n$	Euclidean $n$ -space
$\mathbb{S}^n$	the $n$ -sphere, identified with the set of unit vectors in $\mathbb{R}^{n+1}$
$I$	the unit interval $[0, 1] \subset \mathbb{R}$
$C^1$	continuously differentiable

## Notations defined in the text (with page of first appearance)

Symbol	Description	Page
$\partial X$	the Gromov boundary of $X$ (or the manifold boundary if $X$ is a manifold)	8
$\overline{X}$	$X \cup \partial X$	8
$\prec$	is carried by	34

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$\mathcal{ABL}$	the set of almost boundary laminations	26
$\mathcal{A}(\Sigma)$	the set of isotopy classes of arcs on $\Sigma$	17
$\mathcal{BL}$	the set of boundary laminations	26
$\mathcal{C}(\Sigma)$	the curve graph of $\Sigma$	19
$D_\alpha$	the disc of projective laminations disjoint from $\alpha$	47
$d_\Sigma(\mathfrak{p}, \mathfrak{q})$	the diameter of the set $\pi_\Sigma(\mathfrak{p}) \cup \pi_\Sigma(\mathfrak{q})$ in $\mathcal{C}(\Sigma)$	22
$d_W(\lambda, \mu)$	the diameter of the set $\pi_W(\lambda) \cup \pi_W(\mu)$ in $\mathcal{C}(W)$	45
$d_\alpha(\lambda, \mu)$	the diameter of the set $\pi_\alpha(\lambda) \cup \pi_\alpha(\mu)$ in $\mathcal{F}_\alpha$	92
$\mathcal{EL}(\Sigma)$	the set of ending laminations on $\Sigma$	25
$\mathcal{F}$	the Farey graph	12
$\mathcal{F}_\alpha$	the curve graph of $W_\alpha$	79
$H_\alpha$	the half Dehn twist about $\alpha$	17
$\text{index}(K)$	the Euler index of $K$	34
$\mathcal{L}(\Sigma)$	the set of laminations on $\Sigma$	23
$\text{MCG}(\Sigma)$	the mapping class group of $\Sigma$	17
$\mathcal{ML}(\Sigma)$	the space of measured laminations on $\Sigma$	27
$N_\varepsilon(X)$	an $\varepsilon$ -neighbourhood of $X$	25
$N(\tau)$	a tie neighbourhood of $\tau$	33
$\mathcal{PBL}$	the space of projective boundary laminations	31
$\mathcal{PABL}$	the space of projective almost boundary laminations	31
$\mathcal{PML}(\Sigma)$	the space of projective laminations	29
$\mathcal{P}(\Sigma)$	the pants graph of $\Sigma$	21
$\partial\mathcal{P}$	the Gromov boundary of the pants graph of $\Sigma_{0,5}$	23
$\mathbb{P}(S)$	the power set of $S$	43
$P(\tau)$	the train track polytope of $\tau$	38
$\text{PT } \Sigma$	the projective tangent bundle of $\Sigma$	24
$\mathcal{S}(\Sigma)$	the set of isotopy classes of curves on $\Sigma$	17
$\text{Supp}(\lambda)$	the support of $\lambda$	25
$T_\alpha$	the Dehn twist about $\alpha$	17

$\mathbf{T}_p$	the topology of subsurface projection on $\mathcal{BL}$	78
$\mathbf{T}_s$	the topology of superconvergence	24
$\mathcal{UML}(\Sigma)$	the space of measurable laminations on $\Sigma$	29
$W_\alpha$	the domain in $\Sigma_{0,5}$ bounded by $\alpha$	26
$WP(\Sigma)$	Teichmüller space of $\Sigma$ with the Weil-Petersson metric	81
$\eta$	the injection from $\mathcal{C}(\Sigma_{0,5})$ to $\mathcal{PML}(\Sigma_{0,5})$	46
$\Theta_W$	Klarreich's homeomorphism from $\partial\mathcal{C}(W)$ to $(\mathcal{EL}(W), \mathbf{T}_s)$	31
$\iota(\alpha, \beta)$	the (geometric) intersection number of $\alpha$ and $\beta$	18
$\xi(\Sigma)$	the complexity of $\Sigma$	19
$\pi_W$	subsurface projection to $W$	43
$\pi_\alpha$	subsurface projection to $W_\alpha$	45
$\Pi_W$	the boundary projection to $W$	79
$\Pi_\alpha$	the boundary projection to $W_\alpha$	79
$\Sigma_{g,n}^b$	a surface of genus $g$ with $n$ punctures and $b$ boundary components	16
$\Phi$	the measure-forgetting map from $\mathcal{PML}(\Sigma)$ to $\mathcal{UML}(\Sigma)$	29
$\chi(\Sigma)$	the Euler characteristic of $\Sigma$	17
$\Psi$	the projectivisation map from $\mathcal{ML}(\Sigma)$ to $\mathcal{PML}(\Sigma)$	29
$\Omega$	the map from $\mathcal{PABL}$ to $\mathcal{BL}$	82

### Conventions (with page reference)

1. Surfaces are of finite type and have negative Euler characteristic. 16
2. Curves are isotopy classes of simple closed curves. 17
3. All laminations are geodesic laminations. 23
4. An arc joining two punctures on  $\Sigma_{0,5}$  indicates the curve that bounds a regular neighbourhood of it. 51
5. Neighbourhoods are not required to be open. 107

# Chapter 1

## Introduction

Geodesic laminations were introduced by William P. Thurston in his study of hyperbolic 3-manifolds [Thu1]. There is an intricate relationship between laminations, Teichmüller spaces and mapping class groups. For instance, the set of all laminations, each endowed with the additional structure of a projective transverse measure, forms a boundary for Teichmüller space [Thu4] known as *Thurston's boundary*. The action of the mapping class group on Teichmüller space extends to Thurston's boundary. Over the last 40 years, laminations have become established tools in low-dimensional topology and other areas of mathematics.

To provide a combinatorial framework for studying the action of the mapping class group on Thurston's boundary, Harvey introduced the *curve graph* [Har]. Vertices of this graph correspond to the curves on the surface and, in the generic case, edges correspond to disjointness. Masur and Minsky showed that the curve graph has infinite diameter and is Gromov hyperbolic [MasMi1]. In the celebrated sequel to this publication, the combinatorial and topological properties of the curve graph are used to encode the rich structure of the mapping class group [MasMi2]. Klarreich identified the Gromov boundary of the curve graph with the space of ending laminations [Kla].

The *pants graph* made its first appearance in a paper by Hatcher and Thurston [HatT] in which they prove that the mapping class group is finitely presented. Brock discovered that the pants graph is quasi-isometric to the Teichmüller space with the Weil-Petersson metric [Broc]. The pants graph associated to a surface is Gromov hyperbolic if and only if the surface has complexity at most 2 [BrocF].

Morally, surfaces of low complexity serve as building blocks for surfaces of higher complexity. The curve and pants graphs of any surface of complexity less than 2 are completely characterised – in this sense, complexity-2 surfaces are the simplest

surfaces with ‘interesting’ curve and pants graphs. On the other hand, complexity-2 surfaces are the most complex surfaces whose pants graphs are hyperbolic. There are essentially two surfaces of complexity 2: the twice punctured torus  $\Sigma_{1,2}$  and the five-times punctured sphere  $\Sigma_{0,5}$ . This thesis focuses on the latter.

## 1.1 Structure of the thesis and main results

To guide the reader through the thesis we give a preview of the contents of each chapter, together with the most important results.

### Chapters 2 and 3

The upcoming two chapters are expository and do not present original results. Nevertheless, some of the statements presented in these chapters might not appear in the literature in their current form.

Specifically, in Chapter 2 we define Gromov hyperbolicity and we describe a particular hyperbolic space: the Farey graph. We determine a neighbourhood base for the topology on the Gromov boundary of the Farey graph (Proposition 2.2.8).

Chapter 3 introduces several objects associated to a surface. First of all, we define the curve graph and the pants graph, and state some properties thereof. Subsequently, we give a snapshot of the extensive theory of laminations (a generalisation of curves on a surface) and train tracks (a ‘coordinate system’ for laminations). Finally, we describe how to project curves, laminations and train tracks to a subsurface. This subsurface projection map plays a central role in Chapter 5.

### Chapter 4

In Chapter 4 we study the inclusion of the curve graph  $\mathcal{C}(\Sigma_{0,5})$  into the space of projective laminations  $\mathcal{PML}(\Sigma_{0,5})$ , which is homeomorphic to  $\mathbb{S}^3$  [Thu4]. We endow  $\mathcal{PML}(\Sigma_{0,5})$  with the structure of a  $\Delta$ -complex, obtained by gluing the polytopes of eight complete train tracks (Proposition 4.2.5). Any pentagon in  $\mathcal{C}(\Sigma_{0,5})$  has an associated decagon in  $\mathcal{C}(\Sigma_{0,5})$ , and we prove that the pair includes into  $\mathcal{PML}(\Sigma_{0,5})$  in the following way.

**Proposition 4.3.5.** *The image in  $\mathcal{PML}(\Sigma_{0,5})$  of a pentagon and the associated decagon in  $\mathcal{C}(\Sigma_{0,5})$  is a Hopf link.*

The *pentagon-decagon graph*  $\mathcal{G}_\Gamma$  is the induced subgraph of  $\mathcal{C}(\Sigma_{0,5})$  on a pentagon  $\Gamma$  and the associated decagon (Definition 4.4.1). We utilise the quaternions

to describe a ‘standard’ embedding of this graph into  $\mathbb{S}^3$ . The main result of Chapter 4 can be formulated as follows.

**Theorem 4.4.6.** *The embedding of  $\mathcal{G}_\Gamma$  into  $\mathcal{PML}(\Sigma_{0,5})$  is standard.*

Whilst deferring the details of the standard embedding of  $\mathcal{G}_\Gamma$  to Section 4.4, we use this space to highlight a corollary of Theorem 4.4.6.

**Corollary 4.4.7.** *The image of  $\mathcal{C}(\Sigma_{0,5})$  in  $\mathcal{PML}(\Sigma_{0,5})$  contains a trefoil knot.*

We remark that the results obtained in Chapter 4 have no immediate implications for the subsequent chapters.

Chapters 5, 6 and 7 elaborate on the space of boundary laminations. Here a *boundary lamination* is a minimal lamination on  $\Sigma_{0,5}$  with more than one leaf (Definition 3.2.11).

## Chapter 5

Chapter 5 introduces several topologies on the set of boundary laminations  $\mathcal{BL}$ . Following Brock and Masur [BroCM], we use subsurface projections to define a topology  $\mathbf{T}_p$  on  $\mathcal{BL}$  (Definition 5.1.8). Let  $\mathbf{T}_s$  denote the topology of superconvergence (Definition 3.2.4).

**Proposition 5.1.9.** *The topologies  $\mathbf{T}_p$  and  $\mathbf{T}_s$  on  $\mathcal{BL}$  are the same.*

The neighbourhood base for the boundary of the Farey graph described in Chapter 2 gives rise to a neighbourhood base for  $(\mathcal{BL}, \mathbf{T}_s)$  at every non-filling boundary lamination (Proposition 5.2.4).

Let  $\mathcal{P}(\Sigma)$  denote the pants graph of a surface  $\Sigma$  and write  $\partial X$  for the Gromov boundary of a hyperbolic space  $X$ . Combining Proposition 5.1.9 with a number of known results, we conclude the following.

**Corollary 5.3.5.**  *$\partial\mathcal{P}(\Sigma_{0,5})$  and  $(\mathcal{BL}, \mathbf{T}_s)$  are homeomorphic.*

We finish Chapter 5 with some observations relating  $(\mathcal{BL}, \mathbf{T}_s)$  to  $\mathcal{PML}(\Sigma_{0,5})$ .

## Chapter 6

In Chapter 6 we investigate the connectivity of  $(\mathcal{BL}, \mathbf{T}_s)$ . We prove the following two key results.

**Proposition 6.2.3.**  *$(\mathcal{BL}, \mathbf{T}_s)$  is path connected.*

**Theorem 6.5.3.**  *$(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected.*



The proof of Theorem 6.5.3 relies on Gabai’s result that the space of ending laminations of  $\Sigma_{0,5}$  is locally path connected [Gab1]. Combined with Corollary 5.3.5, these two results have the following implication.

**Corollary 6.5.4.**  *$\partial\mathcal{P}(\Sigma_{0,5})$  is path connected and locally path connected.*

## Chapter 7

Chapter 7 analyses the dimension of  $(\mathcal{BL}, \mathbf{T}_s)$ . We obtain some minor results in this direction. In particular, we observe that the frontiers of the neighbourhoods in  $(\mathcal{BL}, \mathbf{T}_s)$  found in Chapter 5 contain paths (Proposition 7.3.6).

### 1.2 Comparison with previous work

The inclusion of the curve complex into  $\mathcal{PML}(\Sigma)$  was previously studied by Gabai in [Gab2, Gab3]. Motivated by the observation that  $\mathcal{EL}(\Sigma)$  and  $\mathcal{C}(\Sigma)$  ‘almost’ live in  $\mathcal{PML}(\Sigma)$  as complementary objects, Gabai formulates a duality conjecture [Gab2, Conjecture 19.4].

We are not aware of any existing research on knots in the curve graph of  $\Sigma_{0,5}$ . Trying to generalise our observations from Chapter 4 to surfaces of higher complexity, one could – naively – ask whether the curve complex of any surface  $\Sigma$  has a subcomplex homeomorphic to a sphere whose image in  $\mathcal{PML}(\Sigma)$  is knotted. However, if the complexity  $\xi(\Sigma)$  is larger than 2, then this question is easily answered. The top-dimensional simplices of the curve complex of  $\Sigma$  have dimension  $\xi(\Sigma) - 1$ , whereas  $\mathcal{PML}(\Sigma)$  is a sphere of dimension  $2\xi(\Sigma) - 1$  [Thu4]. Since PL spheres of codimension larger than 2 cannot be knotted [Zee], it follows that the image in  $\mathcal{PML}(\Sigma)$  of any sphere in the curve complex is unknotted if  $\xi(\Sigma) > 2$ .

In ‘The PML Visualization Project’ [DuG], Dumas and Guéritaud produced animated images of  $\mathcal{PML}(\Sigma_{0,5})$  using a projection to  $\mathbb{R}^3$ . The vertices of the pentagon and the decagon are clearly visible in this visualisation, and their position seems to suggest that the pentagon and the decagon form a Hopf link. However, the visualisation does not display edges of the curve graph. In fact, Dumas and Guéritaud obtain this projection to  $\mathbb{R}^3$  by composing Thurston’s embedding of  $\mathcal{PML}(\Sigma_{0,5})$  into a cotangent space of Teichmüller space [Thu3] with stereographic projection. This projection does not map edges of the curve graph to linear segments.

Brock and Masur described a homeomorphism between  $(\mathcal{BL}, \mathbf{T}_p)$  and the CAT(0)-boundary of the Weil-Petersson metric on the Teichmüller space of  $\Sigma_{0,5}$  [BroM]. The second space is homeomorphic to  $\partial\mathcal{P}(\Sigma_{0,5})$  [Bro, BuF]. Whereas the

topology on  $\partial\mathcal{P}(\Sigma_{0,5})$  remained relatively unexplored, several authors studied the Gromov boundary of the curve graph. Klarreich proved that  $\partial\mathcal{C}(\Sigma)$  is homeomorphic to the space of ending laminations on  $\Sigma$  [Kla]. Gabai showed that  $\partial\mathcal{C}(\Sigma)$  is a path connected and locally path connected space [Gab1]. Using Gabai's result, Hensel and Przytycki were able to show that  $\partial\mathcal{C}(\Sigma_{0,5})$  is homeomorphic to the Nöbeling curve [HenP]. This result was later generalised by Gabai, who gave a topological characterisation of  $\partial\mathcal{C}(\Sigma)$  in case that  $\Sigma$  is an  $n$ -times punctured sphere, for all  $n \geq 5$  [Gab3].

### 1.3 Future research directions

In the final chapter we formulate a couple of questions related to the dimension of lamination spaces. As far as we are aware, they remain open and could serve as starting points for future research.

*Question 7.2.3.* Is  $\eta(\mathcal{C}) \subset \mathcal{PML}(\Sigma_{0,5})$  locally connected?

If the answer to Question 7.2.3 is negative, then the subspace  $\mathcal{PABL}$  of  $\mathcal{PML}(\Sigma_{0,5})$  must be 2-dimensional.

*Question 7.3.3.* Is  $(\mathcal{BL}, \mathbf{T}_s)$  1-dimensional?

If Question 7.3.3 can be answered affirmatively, a follow-up question could be whether  $(\mathcal{BL}, \mathbf{T}_s)$  is a Nöbeling curve.

## Chapter 2

# Gromov hyperbolic spaces

This background chapter introduces some established concepts in geometric group theory. Section 2.1 defines Gromov hyperbolic spaces and their boundaries, and is only expository. Section 2.2 describes a particular hyperbolic space: the Farey graph.

The main observations of this chapter are Propositions 2.2.7 and 2.2.8. These results are well known among experts. Not being aware of an explicit reference for these statements, we include a proof nevertheless.

### 2.1 Gromov hyperbolicity

Most of the discussion presented in this section can be found in [AIBCFLMSS]. A common reference for Gromov hyperbolicity is Bridson and Haefliger's book [BriH, Chapter III.H]. However, when describing the topology on the Gromov boundary of  $X$ , Bridson and Haefliger assume  $X$  to be a proper metric space. Since we are mainly concerned with locally infinite graphs, we avoid making this assumption.

This section does not present original results.

#### Slim geodesic triangles

Let  $(X, d)$  be a metric space. A *geodesic* in  $X$  is a map  $g$  from a closed, connected set  $R \subset \mathbb{R}$  to  $X$  such that  $d(g(t), g(s)) = |t - s|$  for all  $t, s \in R$ . We often identify  $g$  with its image in  $X$ . Specifically, a geodesic  $g$  is called a *geodesic segment* (respectively *ray*, *line*) if  $R$  is compact (respectively  $R = [0, \infty)$ ,  $R = \mathbb{R}$ ). If any two points in  $X$  can be connected by a geodesic segment we say that  $(X, d)$  is a *geodesic* metric space.

Three geodesic segments  $g_1, g_2, g_3$  in  $X$  form a *geodesic triangle* if, for every  $i \neq j$ , the geodesics  $g_i$  and  $g_j$  have one endpoint in common. Informally speaking, a

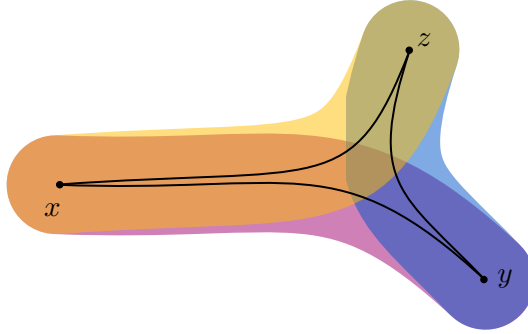


Figure 2.1: A slim geodesic triangle.

geodesic metric space is hyperbolic if every geodesic triangle looks more or less like a ‘tripod’. The following definition makes this precise.

**Definition 2.1.1** (Hyperbolic space). Let  $(X, d)$  be a geodesic metric space and suppose that  $\delta > 0$ . A geodesic triangle in  $X$  consisting of geodesic segments  $g_1, g_2, g_3$  is called  $\delta$ -*slim* if for any  $i$  the geodesic  $g_i$  is contained in a  $\delta$ -neighbourhood of  $\bigcup_{j \neq i} g_j$ .

We say that  $X$  is  $\delta$ -*hyperbolic* if every geodesic triangle in  $X$  is  $\delta$ -slim. Finally,  $X$  is *hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta > 0$ .

*Remark 2.1.2.* Most metric spaces that we will study are (simplicial) graphs. There is a standard way to interpret a graph as a metric space. Let  $X$  be a graph. Parametrise every edge of  $X$  linearly so that it has length 1. Define the distance between two points in  $X$  to be the minimal length of a path connecting them. The resulting metric is called the *graph metric* on  $X$ .

## The Gromov product

Let  $(X, d)$  be a geodesic metric space.

**Definition 2.1.3** (Gromov product). The *Gromov product* of  $x, y \in X$  with respect to  $v \in X$  is

$$(x \cdot y)_v = \frac{1}{2} d(x, v) + d(y, v) - d(x, y).$$

The Gromov product can be used to give a different characterisation of Gromov hyperbolicity [ABCFLMSS, Proposition 2.1 and 2.2].

**Lemma 2.1.4.**  $X$  is hyperbolic if and only if there exists some  $\delta > 0$  such that for all  $x, y, z, v \in X$

$$(x \cdot y)_v \geq \min\{(x \cdot z)_v, (z \cdot y)_v\} - \delta.$$

Note that the constant  $\delta$  in Lemma 2.1.4 does not need to be the same as the constant in Definition 2.1.1. The following lemma shows that, up to bounded error,  $(x \cdot y)_v$  is the distance between the basepoint  $v$  and a geodesic connecting  $x$  and  $y$ . For a proof see, for instance, [Väi, 2.33].

**Lemma 2.1.5.** *Suppose that  $X$  is a hyperbolic space and  $\delta > 0$  is the constant from Lemma 2.1.4. If  $g \subset X$  is a geodesic connecting  $x, y \in X$  and  $z$  is a point on  $g$  closest to  $v$ , then*

$$d(z, v) - 2\delta \leq (x \cdot y)_v \leq d(z, v).$$

## The Gromov boundary

The Gromov boundary is a topological space that encodes the structure of a hyperbolic space ‘at infinity’. Morally, the elements of the Gromov boundary of  $X$  are diverging sequences of points in  $X$ . Two elements in the boundary are close if the diverging sequences stay near each other for a long time.

For the remaining part of this section we assume that  $X$  is a hyperbolic space with a basepoint  $v \in X$  and  $\delta > 0$  is the constant given by Lemma 2.1.4.

A sequence  $(x_i)_{i \in \mathbb{N}}$  in  $X$  *converges at infinity* if  $\lim_{i,j \rightarrow \infty} (x_i \cdot x_j)_v = \infty$ . Observe that this notion is independent of the choice of basepoint. Let  $S_\infty(X)$  be the set of all sequences in  $X$  that converge at infinity. We endow  $S_\infty(X)$  with the equivalence relation given by *fellow-travelling*,

$$(x_i)_{i \in \mathbb{N}} \sim (y_i)_{i \in \mathbb{N}} \iff \lim_{i,j \rightarrow \infty} (x_i \cdot y_j)_v = \infty.$$

**Definition 2.1.6** (Gromov boundary). The *Gromov boundary* of  $(X, d)$  is the set  $\partial X = S_\infty(X) / \sim$ . We write  $\overline{X} = X \cup \partial X$ .

A sequence  $(x_i)_{i \in \mathbb{N}}$  of points in  $X$  *converges to*  $x \in \partial X$  if it converges at infinity and  $x = [(x_i)]$ . In that case we write  $x_i \rightarrow x \in \partial X$ . If  $(x_i)_{i \in \mathbb{N}}$  is a sequence in  $X$  that converges to  $x \in X$  in the ‘usual’ metric sense we also write  $x_i \rightarrow x \in X$ .

In order to describe the topology on  $\overline{X}$ , we extend the definition of the Gromov product to  $\partial X$ .

**Definition 2.1.7** (Extended Gromov product). For  $x, y \in \overline{X}$  define

$$(x \cdot y)_v = \inf \{ \liminf_{i \rightarrow \infty} (x_i \cdot y_i)_v \},$$

where the infimum is taken over all pairs of sequences  $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}$  in  $X$  with the property that  $x_i \rightarrow x$  and  $y_i \rightarrow y$ .

*Remark 2.1.8.* The extended Gromov product extends the Gromov product on  $X$  – hence the name. A proof of this fact is given in [AlBCFLMSS, Lemma 4.5.2].

Informally, the following lemma says that choosing a different representative sequence does not change the extended Gromov product much [AlBCFLMSS, Lemma 4.6.3 and 4.6.4].

**Lemma 2.1.9.** *If  $x, y \in \overline{X}$  and  $x_i \rightarrow x$ ,  $y_i \rightarrow y$  then*

$$(x \cdot y)_v \leq \liminf_{i \rightarrow \infty} ((x_i \cdot y_i)_v) \leq (x \cdot y)_v + 2\delta.$$

A geodesic ray  $g$  connects  $x \in X$  and  $y \in \partial X$  if  $g(0) = x$  and  $g(n) \rightarrow y$  as  $n \rightarrow \infty$ . Similarly, a geodesic line  $g$  connects  $x, y \in \partial X$  if  $g(-n) \rightarrow x$  and  $g(n) \rightarrow y$  as  $n \rightarrow \infty$ . We generalise Lemma 2.1.5 to  $\overline{X}$  as follows.

**Lemma 2.1.10.** *Suppose that  $g$  is a geodesic connecting  $x, y \in \overline{X}$ . If  $z \in X$  is a point on  $g$  closest to  $v$ , then*

$$d(z, v) - 4\delta \leq (x \cdot y)_v \leq d(z, v).$$

*Proof.* Let  $x, y, z$  and  $g$  be as in the statement. Up to reparametrising, we may assume that  $g(0) = z$ . For  $i \in \mathbb{N}$  define  $y_i = y$  if  $y \in X$  and define  $y_i = g(i)$  otherwise. Similarly, set  $x_i = x$  if  $x \in X$  and  $x_i = g(-i)$  otherwise. Observe that  $x_i \rightarrow x$  and  $y_i \rightarrow y$ . Lemma 2.1.5 gives that

$$d(z, v) - 2\delta \leq (x_i \cdot y_i)_v \leq d(z, v)$$

for all  $i \in \mathbb{N}$ . Taking the infimum limit we obtain

$$d(z, v) - 2\delta \leq \liminf_{i \rightarrow \infty} (x_i \cdot y_i)_v \leq d(z, v).$$

With Lemma 2.1.9 this implies  $d(z, v) - 4\delta \leq (x \cdot y)_v \leq d(z, v)$ . □

*Remark 2.1.11.* If  $X$  is a proper hyperbolic space then any two points in  $\overline{X}$  can be connected by a geodesic [BriH, Lemmas 3.1 and 3.2]. This is not necessarily true for arbitrary hyperbolic spaces [KaB, Remark 2.16].

## The topology on the Gromov boundary

We use the extended Gromov product to describe a topology on  $\partial X$  and  $\overline{X}$ . Suppose that  $n \in \mathbb{N}$ . For  $x \in \partial X$ , let  $B(x, n) \subset \overline{X}$  consist of all  $y \in \overline{X}$  with  $(x \cdot y)_v > n$ . For

$x \in X$ , define  $B(x, n) = \{y \in X : d(x, y) < 1/n\}$ . The collection

$$\mathcal{B} = \{B(x, n) : x \in \overline{X}, n \in \mathbb{N}\}$$

is a base for a topology on  $\overline{X}$  [AlBCFLMSS, Proposition 4.8]. Definitions of base and neighbourhood base at a point are included in Appendix A, see also [Wil, §2.4–2.5].

**Definition 2.1.12** (Gromov topology). We endow the set  $\overline{X}$  with the topology associated to the base  $\mathcal{B}$ . This topology on  $\overline{X}$  and its restriction to  $\partial X$  are called the *Gromov topology*.

From now on  $\overline{X}$  and  $\partial X$  will always denote these topological spaces.

*Remark 2.1.13.* The collection

$$\mathcal{B}_x = \{B(x, n) : n \in \mathbb{N}\}$$

is an open neighbourhood base at  $x$  for the Gromov topology on  $\overline{X}$ . This is a consequence of Theorem A.4 and the following fact. For all  $x, y \in \overline{X}$  and  $m \in \mathbb{N}$  with  $x \in B(y, m)$  there exists  $n \in \mathbb{N}$  such that  $B(x, n) \subset B(y, m)$  [AlBCFLMSS, proof of Proposition 4.8].

We list a few properties of the Gromov boundary. Proofs of these facts can be found in [Gro]. The space  $\overline{X}$  is connected and contains  $X$  (with the metric topology) as a dense open subspace. The isometry group of  $X$  acts on  $\partial X$  by homeomorphisms. More generally, every isometric map  $X \rightarrow Y$  of hyperbolic spaces extends to a unique topological embedding  $\overline{X} \rightarrow \overline{Y}$ . Gromov proved that  $\partial X$  is a completely metrisable Hausdorff space [Gro, 1.8B]. Therefore, the topology on  $\partial X$  is determined by its convergent sequences. These are exactly the sequences  $(x_i)_{i \in \mathbb{N}}$  of points in  $\partial X$  for which there exists an  $x \in \partial X$  such that  $(x_i \cdot x)_v \rightarrow \infty$ .

## 2.2 The Farey graph

This section presents an example of a graph that is a hyperbolic metric space. The Farey graph has been intensively studied and it is linked to several areas of mathematics. We will encounter the Farey graph again in the next chapter, as the most elementary curve graph and therefore as a subgraph of every pants graph.

The main goal of this section is to prove Proposition 2.2.8. This gives an explicit neighbourhood base at every boundary point of the Farey graph. Along the way we prove Proposition 2.2.7, which is also of individual interest.

## A combinatorial definition of the Farey graph

The Farey graph is often defined using continued fractions. A detailed account on this can be found, for instance, in [Hat3, Chapter 1]. We prefer to define the Farey graph purely as a combinatorial object.

**Lemma 2.2.1.** *There is a unique non-empty connected simplicial 2-complex  $X$  satisfying the following conditions.*

(C1) *The link of every vertex of  $X$  is homeomorphic to  $\mathbb{R}$ .*

(C2) *The complement of every 1-simplex has two connected components.*

*Proof.* Suppose that  $X$  is a simplicial complex satisfying (C1) and (C2). An example of a complex with these properties is given in Figure 2.2, proving existence.

For the proof of uniqueness we introduce ‘triangle paths’. A *triangle path* is a sequence of distinct 2-simplices  $\Delta_0, \Delta_1, \dots, \Delta_n$  of  $X$  with the property that any  $\Delta_i$  and  $\Delta_{i+1}$  have a common face of dimension 1.

**Claim.** *Any two 2-simplices  $\Delta, E$  of  $X$  can be joined by a unique triangle path.*

*Proof of claim.* To prove the claim, choose interior points  $x$  of  $\Delta$  and  $y$  of  $E$ . Since  $X$  is connected and satisfies condition (C1), there exists a piecewise linear path from  $x$  to  $y$  that is disjoint from  $X^0$ . This path intersects only finitely many simplices, and we may assume that it enters and exits any simplex at most once. The sequence of 2-simplices that this path passes through is the required triangle path. It follows from condition (C2) that such a sequence is unique. This proves the claim.  $\square$

Suppose that  $X, Y$  are simplicial 2-complexes both satisfying (C1) and (C2). We define a simplicial isomorphism  $f: X \rightarrow Y$  by induction of the length of triangle paths. Let  $\Delta$  be a 2-simplex of  $X$  and let  $\Delta'$  be a 2-simplex of  $Y$ . Choose an identification of the vertices of  $\Delta$  with the vertices of  $\Delta'$  and define  $f(\Delta) = \Delta'$  accordingly. This is the base case. For the induction step, suppose that  $f$  is defined for all 2-simplices of  $X$  that can be connected to  $\Delta$  by a triangle path of length  $n$ . Let  $E$  be a 2-simplex with a triangle path  $\Delta = \Delta_0, \Delta_1, \dots, \Delta_n, E$  of length  $n + 1$ . Write  $d$  for the 1-simplex that is a common face of  $\Delta_n$  and  $E$ . Extend  $f$  to  $E$  in such a way that  $f(E)$  is a 2-simplex  $E'$  of  $Y$  that is different from  $f(\Delta_n)$  and has  $f(d)$  as a face. Such a 2-simplex  $E'$  exists and is unique, because  $Y$  satisfies (C1). Since every 2-simplex of  $X$  can be connected to  $\Delta$  by a unique triangle path, we obtain a well defined simplicial map  $f: X \rightarrow Y$ . Bijectivity follows from the existence and uniqueness of triangle paths in  $Y$ .  $\square$



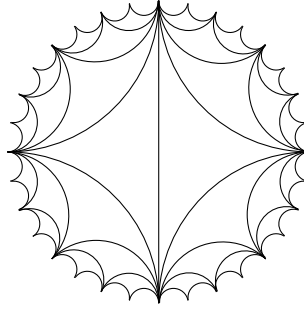


Figure 2.2: A finite subgraph of the Farey graph.

**Definition 2.2.2.** A simplicial 2-complex satisfying the conditions of Lemma 2.2.1 is called the *Farey complex*. The 1-skeleton of the Farey complex is called the *Farey graph*, denoted  $\mathcal{F}$ .

Note that the Farey complex is a flag complex: it is the maximal simplicial complex that has the Farey graph as its 1-skeleton. The Farey graph becomes a metric space with the graph metric (see Remark 2.1.2). With this metric  $\mathcal{F}$  is hyperbolic. This is a well known fact, proved, for instance, in [Mins2, §3]. The Farey graph can be drawn conveniently as a tessellation of  $\mathbb{H}^2$  by ideal triangles, see Figure 2.2.

Write  $V(\mathcal{F})$  and  $E(\mathcal{F})$  for the sets of vertices and edges of  $\mathcal{F}$ , respectively. As a convention, every edge is closed as a subset of  $\mathcal{F}$ , that is, it includes its endpoints.

### Separating edges

The remaining part of this section, which contains Propositions 2.2.8 and 2.2.7, is inspired by Minsky's discussion of separating edges in the Farey graph [Mins2, §3]. However, Minsky does not mention these statements.

For  $U \subset \mathcal{F}$ , let  $\partial U$  be the set of all  $x \in \partial \mathcal{F}$  for which there exist  $x_i \in U$  such that  $x_i \rightarrow x$ . Write  $\bar{U} = U \cup \partial U$ .

**Lemma 2.2.3.** *If  $U$  and  $V$  are the components of  $\mathcal{F} - e$  for some  $e \in E(\mathcal{F})$ , then  $\partial U \cap \partial V = \emptyset$ . Furthermore,  $\bar{U}$  and  $\bar{V}$  are open connected subsets of  $\bar{\mathcal{F}}$ .*

*Proof.* Suppose that  $e \in E(\mathcal{F})$  and let  $U$  and  $V$  be the components of  $\mathcal{F} - e$ . Fix a basepoint  $v \in \mathcal{F}$  and define  $M = \max_{z \in e} d(v, z)$ .

We first prove that  $\partial U \cap \partial V = \emptyset$ . Suppose that  $(x_i)_{i \in \mathbb{N}}$  and  $(y_i)_{i \in \mathbb{N}}$  are sequences that converge at infinity, such that  $x_i \in U$  and  $y_i \in V$  for every  $i \in \mathbb{N}$ . For

$i, j \in \mathbb{N}$ , any geodesic from  $x_i$  to  $y_j$  intersects  $e$ . Hence  $(x_i \cdot y_j)_v \leq M$  by Lemma 2.1.5. This shows that if  $x_i \rightarrow x$  and  $y_i \rightarrow y$  then  $x \neq y$ .

We now prove that  $\overline{U}$  is connected and open – it then follows by symmetry that  $\overline{V}$  is connected and open as well. To see that  $\overline{U}$  is connected, it suffices to note that  $U \subset \overline{U}$  is a connected dense subset.

To show that  $\overline{U}$  is open in  $\overline{\mathcal{F}}$ , we prove that it contains a neighbourhood of each of its points. If  $x \in U \subset \overline{U}$  then  $U$  is an open neighbourhood of  $x$  in  $\overline{\mathcal{F}}$ . This follows from the fact that  $U$  is open in  $\mathcal{F}$  and  $\mathcal{F}$  is open in  $\overline{\mathcal{F}}$ . If  $x \in \partial U \subset \overline{U}$  we claim that the open neighbourhood  $B(x, M)$  of  $x$  (see Remark 2.1.13) is contained in  $\overline{U}$ . Suppose that  $y \in \overline{\mathcal{F}}$  with  $(x \cdot y)_v > M$ . Let  $(x_i)_{i \in \mathbb{N}}$  be a sequence in  $U$  that converges to  $x$ . Take a sequence  $(y_i)_{i \in \mathbb{N}}$  in  $\mathcal{F}$  converging to  $y$  (if  $y \in \mathcal{F}$  take the constant sequence  $y_i = y$  for every  $i \in \mathbb{N}$ ). By Lemma 2.1.9,  $\liminf_{i \rightarrow \infty} (x_i \cdot y_i)_v > M$ . Therefore  $(x_i \cdot y_i)_v > M$  for all sufficiently large  $i$ . By Lemma 2.1.5, the distance from  $v$  to any point on a geodesic connecting  $x_i$  and  $y_i$  is larger than  $M$ . In particular,  $x_i$  and  $y_i$  can be connected by a path in  $\mathcal{F}$  disjoint from  $e$ . This shows that  $(y_i)$  is eventually contained in  $U$ , which proves that  $y \in \overline{U}$ . Hence  $B(x, M) \subset \overline{U}$ .  $\square$

As a consequence of Lemma 2.2.3, the complement of any  $e \in E(\mathcal{F})$  in  $\overline{\mathcal{F}}$  has two connected components.

**Definition 2.2.4** (Separating edge). An edge  $e \in E(\mathcal{F})$  *separates*  $x, y \in \overline{\mathcal{F}} - e$  if  $x$  and  $y$  are contained in distinct components of  $\overline{\mathcal{F}} - e$ . The set of all edges that separate  $x$  and  $y$  is denoted  $E(x, y) \subset E(\mathcal{F})$ .

For  $e, e' \in E(x, y)$ , we write  $e < e'$  if and only if  $e$  intersects the component of  $\overline{\mathcal{F}} - e'$  that contains  $x$ . This defines a total order on  $E(x, y)$ . We write

$$E(x, y) = \{e_i\}_{i \in J}$$

with  $e_i < e_{i+1}$  for all  $i \in J$ . Here  $J$  is the totally ordered set  $\{1, \dots, n\}, \mathbb{N}$  or  $\mathbb{Z}$  depending on whether  $x, y \in \mathcal{F}$ ,  $x \in \mathcal{F}$  and  $y \in \partial \mathcal{F}$  or  $x, y \in \partial \mathcal{F}$ . Observe that  $\text{diam}(e_i \cup e_{i+1}) = 1$ , in other words, the induced subgraph on the vertices of  $e_i$  and  $e_{i+1}$  is a triangle. An example is given in Figure 2.3.

*Remark 2.2.5.* Suppose that  $X$  and  $Y$  are subsets of  $V(\mathcal{F})$  of diameter at most 1. There exist  $x \in X$  and  $y \in Y$  such that  $E(x', y') \subset E(x, y)$  for all  $x' \in X$ ,  $y' \in Y$ . In this case, we write  $E(X, Y) = E(x, y)$ .

We make the following observation.

**Lemma 2.2.6.** *Suppose that  $e \in E(\mathcal{F})$ . A geodesic connecting  $x, y \in \overline{\mathcal{F}}$  intersects both components of  $\mathcal{F} - e$  if and only if  $e \in E(x, y)$ .*

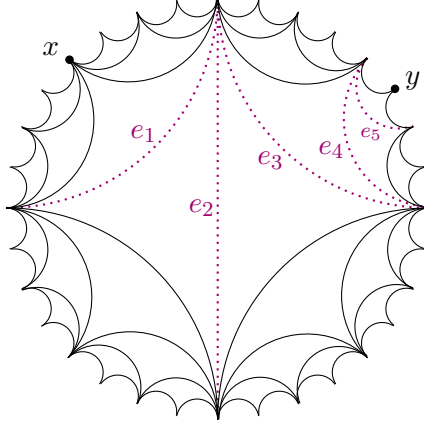


Figure 2.3: Each of dotted edges separates the vertices  $x, y$  of  $\mathcal{F}$ .

*Proof.* Suppose that  $e \in E(\mathcal{F})$  and  $g$  is a geodesic in  $\mathcal{F}$  connecting  $x, y \in \overline{\mathcal{F}}$ . Choose  $s_i, t_i \in \mathbb{R}$  such that  $g(s_i) \rightarrow x$  and  $g(t_i) \rightarrow y$ . If  $e \in E(x, y)$  then, whenever  $i \in \mathbb{N}$  is sufficiently large,  $g(s_i)$  and  $g(t_i)$  are contained in distinct components of  $\mathcal{F} - e$ . This proves that  $g$  intersects both components of  $\mathcal{F} - e$ .

On the other hand, if  $e \notin E(x, y)$  and  $g$  intersects both components of  $\mathcal{F} - e$ , then  $g$  restricts to a path in  $\mathcal{F} - e$  with endpoints on  $e$ . Replacing this path by a path that stays in  $e$  we obtain a shortcut of  $g$ . This contradicts that  $g$  is a geodesic. We conclude that if  $g$  intersects both components of  $\mathcal{F} - e$  then  $e \in E(x, y)$ .  $\square$

Despite being non-proper (compare Remark 2.1.11), the Farey graph has the following property.

**Proposition 2.2.7.** *Any  $x, y \in \overline{\mathcal{F}}$  can be connected by a geodesic.*

*Proof.* Suppose that  $x, y \in \partial\mathcal{F}$  (the proof in case  $x \in \mathcal{F}$  and  $y \in \partial\mathcal{F}$  is similar). Let  $E(x, y) = \{e_i\}_{i \in \mathbb{Z}}$  be the set of edges separating  $x$  and  $y$ . It follows from Lemma 2.2.6 that the endpoints of  $e_i$  converge to  $y$  as  $i \rightarrow \infty$  and to  $x$  as  $i \rightarrow -\infty$ . For every  $i \in \mathbb{N}$  let  $\Gamma_i$  be the induced subgraph of  $\mathcal{F}$  on the vertices of  $e_{-i}, e_{-i+1}, \dots, e_{i-1}, e_i$ . By Lemma 2.2.6, any geodesic segment in  $\mathcal{F}$  that connects an endpoint of  $e_{-i}$  to an endpoint of  $e_i$  is contained in  $\Gamma_i$ . Note that there are only finitely many such segments, since  $\Gamma_i$  is a finite graph.

We will inductively define a sequence  $(g_i)_{i \in \mathbb{N}}$  of geodesic segments in  $\mathcal{F}$  such that every  $g_i$  connects  $e_{-i}$  to  $e_i$  and  $g_j \cap \Gamma_i = g_i$  for all  $j \geq i$ . For any  $i \in \mathbb{N}$  let  $g'_i$  be any geodesic segment from an endpoint of  $e_{-i}$  to an endpoint of  $e_i$ . Fix  $i \in \mathbb{N}$ . If  $j \geq i$ , then  $g'_j$  intersects  $\Gamma_i$  in a geodesic connecting some endpoint of  $e_{-i}$  to some endpoint of  $e_i$ . Since there are only finitely many such segments, up to taking a subsequence every  $g'_j$  intersect  $\Gamma_i$  in the same subgeodesic. Define  $g_i$  to be that

geodesic. We may take a further subsequence so that every  $g'_j$  with  $j \geq i+1$  intersects  $\Gamma_{i+1}$  in the same geodesic segment, denoted by  $g_{i+1}$ . By induction, we obtain a sequence  $(g_i)_{i \in \mathbb{N}}$  of geodesic segments in  $\mathcal{F}$  with the property that  $g_j \cap \Gamma_i = g_i$  for all  $j \geq i$ , such that the endpoints of  $g_i$  converge to  $x$  and  $y$  (respectively) as  $i \rightarrow \infty$ .

Now  $g = \bigcup_{i \in \mathbb{N}} g_i$  is a geodesic from  $x$  to  $y$ .  $\square$

## The boundary of the Farey graph

We find the following characterisation of the topology on  $\partial\mathcal{F}$ .

**Proposition 2.2.8.** *Suppose that  $x \in \partial\mathcal{F}$ . For every  $e \in E(\mathcal{F})$ , let  $A(x, e)$  be the component of  $\overline{\mathcal{F}} - e$  that contains  $x$ . The collection*

$$\mathcal{A}_x = \{A(x, e) : e \in E(\mathcal{F})\}$$

*is an open neighbourhood base at  $x$  for the topology on  $\overline{\mathcal{F}}$ .*

*Proof.* Fix  $x \in \partial\mathcal{F}$ . By Lemma 2.2.3 every  $A(x, e)$  is open in  $\overline{\mathcal{F}}$ . We need to show that, for every open set  $U \subset \overline{\mathcal{F}}$  containing  $x$ , there is some  $e \in E(\mathcal{F})$  such that  $A(x, e) \subset U$ . By Remark 2.1.13, it suffices to prove this for  $U \in \mathcal{B}_x$ .

Choose  $\delta > 0$  as in Lemma 2.1.4 for  $X = \mathcal{F}$ . Fix a basepoint  $v \in V(\mathcal{F})$ . Assume that  $U = \{y \in \overline{\mathcal{F}} : (x \cdot y)_v > n\}$  for some  $n \in \mathbb{N}$ . Take an edge  $e \in E(v, x)$  such that  $\min_{z \in e} d(z, v) > n + 4\delta$ . We will show that  $A(x, e) \subset U$ . Suppose that  $y \in A(x, e)$ . By Proposition 2.2.7 there exists a geodesic from  $x$  to  $y$ . By Lemma 2.2.6, this geodesic is contained in  $A(x, e) \cup e$ . Lemma 2.1.10 gives that  $(x \cdot y)_v > n$ , hence  $y \in U$ .  $\square$

Recall that a topological space is *totally disconnected* if its connected components are points.

**Lemma 2.2.9.**  *$\partial\mathcal{F}$  is totally disconnected.*

*Proof.* Suppose that  $x, y \in \partial\mathcal{F}$  and  $e \in E(x, y)$ . Lemma 2.2.3 shows that  $A(x, e)$  and  $A(y, e)$  (defined as in Proposition 2.2.8) are disjoint open subsets of  $\overline{\mathcal{F}}$ . Since  $\partial\mathcal{F} \subset A(x, e) \cup A(y, e)$ , this shows that  $x$  and  $y$  are not contained in the same component of  $\partial\mathcal{F}$ .  $\square$

In fact, identifying the vertices of  $\mathcal{F}$  with the rational numbers (see, for instance, [Hat3, Chapter 1]) one finds an explicit homeomorphism  $\partial\mathcal{F} \cong \mathbb{R} - \mathbb{Q}$ .

## Chapter 3

# Surfaces

The purpose of this chapter is to introduce several spaces associated to a surface, each with an induced action of the mapping class group.

Section 3.1 discusses curves and pants decompositions. In Section 3.2 we introduce laminations and see that  $\text{MCG}(\Sigma)$  acts by homeomorphisms on certain spaces of those. Section 3.3 is about train tracks, a way of assigning coordinates to laminations. Lastly, we talk about subsurface projection in Section 3.4.

This chapter does not present original results.

### 3.1 Curves and isotopy

We will recall the definition of the mapping class group, the curve graph and the pants graph. The mapping class group acts on either graph by graph automorphisms.

#### Mapping classes

A *surface* is a manifold of real dimension 2. Denote by  $\Sigma_{g,n}^b$  a connected oriented surface of genus  $g$  with  $n$  punctures and  $b$  boundary components. We always assume that  $g, n$  and  $b$  are finite – that is, all surfaces are of *finite type*. When  $n$  or  $b$  equals 0 we omit it from the notation and write  $\Sigma_g^b$  or  $\Sigma_{g,n}$ , respectively. We refer to the boundary components as *holes*. For instance,  $\Sigma_{0,n}$  is an  $n$ -times punctured sphere and  $\Sigma_0^b$  is a  $b$ -holed sphere. By the classification theorem of surfaces, every connected orientable surface is homeomorphic to exactly one  $\Sigma_{g,n}^b$ . Observe that the interior of  $\Sigma_{g,n}^b$  is homeomorphic to  $\Sigma_{g,n+b}$ . Let

$$\chi(\Sigma_{g,n}^b) = 2 - 2g - n - b$$

denote the *Euler characteristic* of  $\Sigma_{g,n}^b$ . We only consider surfaces of negative Euler characteristic.

For a surface  $\Sigma$  with boundary  $\partial\Sigma$ , let  $\text{Homeo}^+(\Sigma)$  denote the group of orientation preserving homeomorphisms of  $\Sigma$ . Let  $\text{Homeo}^+(\Sigma, \partial\Sigma) < \text{Homeo}^+(\Sigma)$  be the subgroup consisting of all homeomorphisms that fix every component of  $\partial\Sigma$  pointwise and let  $\text{Homeo}_0(\Sigma) < \text{Homeo}^+(\Sigma)$  consist of all homeomorphisms isotopic to the identity map. Write  $\text{Homeo}_0(\Sigma, \partial\Sigma) = \text{Homeo}_0(\Sigma) \cap \text{Homeo}^+(\Sigma, \partial\Sigma)$ . The quotient

$$\text{MCG}(\Sigma) = \text{Homeo}^+(\Sigma, \partial\Sigma) / \text{Homeo}_0(\Sigma, \partial\Sigma)$$

defines a group, called the *mapping class group* of  $\Sigma$ . Elements of  $\text{MCG}(\Sigma)$  are called *mapping classes*. The mapping class group is one of the main objects of study in geometric group theory. Allowing also orientation-reversing homeomorphisms we find a group  $\text{MCG}^\pm(\Sigma)$  that contains  $\text{MCG}(\Sigma)$  as an index-2 subgroup, called the *extended mapping class group* of  $\Sigma$ .

We set up some terminology about curves. A *simple closed curve*  $a$  on a surface  $\Sigma$  is an embedded circle in the interior of  $\Sigma$ . Two simple closed curves  $a$  and  $b$  are *isotopic* if there is some  $f \in \text{Homeo}_0(\Sigma)$  with  $f(a) = b$ . We say that  $a$  is *essential* if it does not bound a disc or a once punctured disc, and *peripheral* if it bounds an annulus. Write  $\mathcal{S}(\Sigma)$  for the set of isotopy classes of essential, non-peripheral simple closed curves on  $\Sigma$ . For the sake of brevity we call elements of  $\mathcal{S}(\Sigma)$  simply ‘curves’. When it is important to distinguish between representatives and equivalence classes, we will use Roman letters for representative curves and the corresponding Greek letters for their isotopy classes. For example,  $[a] = \alpha$  and  $[b] = \beta$ .

An *arc* on  $\Sigma$  is the image of an embedding  $p: I \rightarrow \Sigma$ . Say that an arc is *proper* if  $p^{-1}(\partial\Sigma) = \{0, 1\}$ . Two proper arcs  $\mathbf{a}$  and  $\mathbf{b}$  are *isotopic* if there is some  $f \in \text{Homeo}_0(\Sigma)$  with  $f(\mathbf{a}) = \mathbf{b}$ . A proper arc is *essential* if its complement does not contain a disc or a once punctured disc. Write  $\mathcal{A}(\Sigma)$  for the set of isotopy classes of essential proper arcs on  $\Sigma$ .

*Example 3.1.1.* A well-known example of a mapping class is the *Dehn twist*  $T_\alpha$  about a curve  $\alpha$ . The Dehn twist is represented by a homeomorphism that differs from the identity only in an annular neighbourhood of the curve, where it ‘twists’ in the direction indicated by the orientation on the surface. The effect of  $T_\alpha$  on an arc intersecting  $\alpha$  is illustrated in Figure 3.1. If  $\alpha$  bounds a twice punctured disc  $D$ , the *half Dehn twist*  $H_\alpha$  is the isotopy class of a homeomorphism that differs from the identity only on  $D$ , where it ‘twists’ according to the orientation of the surface, see Figure 3.2. Note that  $H_\alpha^2 = T_\alpha$ .

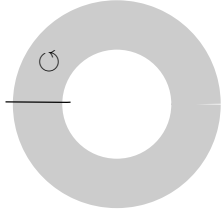


Figure 3.1: The Dehn twist.

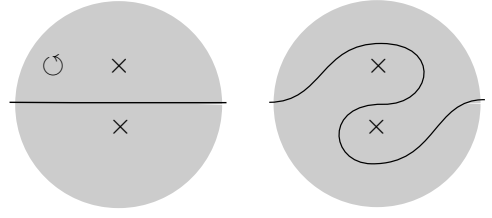


Figure 3.2: The half Dehn twist.

The following theorem affirms the importance of Dehn twists. The case  $b = 0$  was proven by Dehn in the 1920s [Deh2]. Lickorish later presented an independent proof, giving explicit generators [Lic]. An account on the general case can be found in [FarM, Theorem 4.11].

**Theorem 3.1.2** (Dehn, Lickorish).  $\text{MCG}(\Sigma_g^b)$  is generated by finitely many Dehn twists.

**Corollary 3.1.3.**  $\text{MCG}(\Sigma_{g,n}^b)$  is generated by finitely many Dehn twists and half Dehn twists.

*Proof.* Consider the homomorphism from  $\text{MCG}(\Sigma_{g,n}^b)$  to the symmetric group on  $n$  elements, that assigns to a mapping class the induced permutation on the punctures of  $\Sigma$ . This homomorphism is surjective and its kernel is  $\text{MCG}(\Sigma_g^{b+n})$ , which is finitely generated by Dehn twists. Observe that every transposition of the punctures can be realised by a half Dehn twist. Since the symmetric group is finite and generated by transpositions, the result follows.  $\square$

### The curve graph

We structure the infinite set of curves  $\mathcal{S}(\Sigma)$  by viewing it as the vertex set of a metric graph. To describe the edge set we need to talk about intersections of curves. The *intersection number* of  $\alpha, \beta \in \mathcal{S}(\Sigma) \cup \mathcal{A}(\Sigma)$  is defined as

$$\iota(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

Note that the intersection number of any two curves or arcs is a non-negative integer. Two representatives  $a \in \alpha$  and  $b \in \beta$  *intersect minimally* if they are transverse and  $\iota(\alpha, \beta) = |a \cap b|$ . When  $\iota(\alpha, \beta) = 0$  we say that  $\alpha$  and  $\beta$  are *disjoint* and when  $\iota(\alpha, \beta) = k$  we say that  $\alpha$  and  $\beta$  *intersect  $k$  times*. Recall that two embedded curves  $a$  and  $b$  in  $\Sigma$  are said to *form a bigon* if there is an embedded disc in  $\Sigma$  whose boundary is the union of an arc in  $a$  and an arc in  $b$  intersecting in exactly two

points. The following proposition relates this property to minimal intersection [FarM, Proposition 1.7].

**Lemma 3.1.4** (Bigon criterion). *Two simple closed curves  $a$  and  $b$  intersect minimally if and only if they do not form a bigon.*

A *multicurve* is a set  $\{\alpha_1, \dots, \alpha_n\} \subset \mathcal{S}(S)$  such that  $\iota(\alpha_i, \alpha_j) = 0$  for all  $i \neq j$ . The *complexity* of the surface  $\Sigma_{g,n}^b$  is the number

$$\xi(\Sigma_{g,n}^b) = 3g - 3 + n + b.$$

This is exactly the number of curves that can be simultaneously realised disjointly. In other words,  $\xi(\Sigma)$  is the maximal number of elements in a multicurve on  $\Sigma$ .

**Definition 3.1.5** (Curve graph,  $\xi \geq 2$ ). Let  $\Sigma$  be a surface with  $\xi(\Sigma) \geq 2$ . The *curve graph*  $\mathcal{C}(\Sigma)$  is a graph with vertex set  $\mathcal{S}(\Sigma)$ . Edges of  $\mathcal{C}(\Sigma)$  correspond to pairs of distinct disjoint curves.

In 1981, Harvey introduced the curve graph and observed that it is connected [Har, Proposition 2]. The simplicial complex obtained by adding a simplex to  $\mathcal{C}(\Sigma)$  for each multicurve in  $\Sigma$  is known as the *curve complex*. Note that the curve complex of  $\Sigma$  has dimension  $\xi(\Sigma) - 1$  and its 1-skeleton is the curve graph.

*Remark 3.1.6.* Every vertex  $\alpha$  of the curve graph has infinite valence. To see this, let  $\beta, \gamma$  be two curves in the complement of  $\alpha$  such that  $\iota(\beta, \gamma) > 0$ . Then  $T_\gamma^k(\beta) \neq T_\gamma^m(\beta)$  whenever  $k \neq m$ , since  $\iota(T_\gamma^k(\beta), \beta) = |k|\iota(\gamma, \beta)^2$  [FarM, Proposition 3.2]. However,  $\iota(\alpha, T_\gamma^k(\beta)) = 0$  for any  $k$ , so there are infinitely many distinct vertices that are connected to  $\alpha$  by an edge.

We can also define curve graphs for surfaces of complexity less than 2. This graph will have the same vertex set  $\mathcal{S}(\Sigma)$ . However, a surface of complexity less than 2 has no pair of disjoint essential non-peripheral simple closed curves and therefore the graph of Definition 3.1.5 would be totally disconnected. For  $k \geq 0$  define a set of edges  $E_k(\Sigma)$  by

$$\{\alpha, \beta\} \in E_k(\Sigma) \iff \iota(\alpha, \beta) \leq k,$$

where  $\alpha, \beta \in \mathcal{S}(\Sigma)$ . Define the  *$k$ -curve graph* to be the graph  $\mathcal{C}_k(\Sigma)$  with vertex set  $\mathcal{S}(\Sigma)$  and edge set  $E_k(\Sigma)$ . Note that when  $\xi(\Sigma) \geq 2$  the graph  $\mathcal{C}_0(\Sigma)$  is exactly the curve graph of  $\Sigma$  from Definition 3.1.5. We use  $\mathcal{C}_k(\Sigma)$  to generalise the definition of the curve graph to surfaces of any complexity.



**Definition 3.1.7** (Curve graph). The *curve graph* of a surface  $\Sigma$  is the graph  $\mathcal{C}(\Sigma) = \mathcal{C}_k(\Sigma)$ , where  $k$  is the smallest integer such that  $\iota(\alpha, \beta) = k$  for some  $\alpha, \beta \in \mathcal{S}(\Sigma)$ .

We say that a surface is a *pair of pants* if its interior is a thrice punctured sphere. That is, a pair of pants can be any of the surfaces  $\Sigma_{0,3}$ ,  $\Sigma_{0,2}^1$ ,  $\Sigma_{0,1}^2$  and  $\Sigma_0^3$ . Note that the curve graph of a surface with boundary is the same as the curve graph of its interior, so it suffices to discuss curve graphs of compact surfaces.

We will say a few words about each of the low complexity cases separately. The only compact surface satisfying  $\chi(\Sigma) < 0$  and  $\xi(\Sigma) = 0$  is the 3-holed sphere. A pair of pants has no essential curves and therefore its curve graph is trivial. The compact surfaces of complexity 1 are the one-holed torus and the 4-holed sphere. Cutting the once holed torus along a simple closed curve we obtain a pair of pants. It is easy to find two curves on the one-holed torus that intersect once. This shows that  $k = 1$  in this case. Any essential non-peripheral curve cuts  $\Sigma_0^4$  into two pairs of pants and  $\Sigma_0^4$  is a planar surface, so there are no curves that intersect once. This shows that  $k = 2$ .

*Remark 3.1.8.* It is well known that the curve graphs of  $\Sigma_1^1$  and  $\Sigma_0^4$  are isomorphic to the Farey graph [HatT, BowE]. See also [FarM, §4.1].

The extended mapping class group acts on the curve graph in the following way. For  $[f] \in \text{MCG}^\pm(\Sigma)$ , define

$$[f](\alpha) = [fa],$$

where  $\alpha = [a] \in \mathcal{S}(\Sigma)$ . This assignment is well defined. Furthermore,  $[f]$  preserves the intersection number, so it maps edges of  $\mathcal{C}_k(\Sigma)$  to edges. This defines a homomorphism from  $\text{MCG}^\pm(\Sigma)$  to the automorphism group of  $\mathcal{C}_k(\Sigma)$ . Ivanov proved that this homomorphism is an isomorphism [Iva3, Kor, Luo].

*Remark 3.1.9.* The homomorphism  $\text{MCG}(\Sigma_0^4) \rightarrow \text{Aut}(\mathcal{F})$  is surjective. Consequently, if  $\Sigma$  is a surface whose interior is a 4-times punctured sphere, then every automorphism of  $\mathcal{C}(\Sigma)$  is induced by an element of  $\text{MCG}(\Sigma)$ .

The graph metric makes the curve graph into a geodesic metric space  $(\mathcal{C}(\Sigma), d_\Sigma)$ . A surgery argument shows that the distance  $d_\Sigma$  is bounded by a linear function of the intersection number [MasMi1, Lemma 2.1]. Bowditch found the following bound [Bow2, Lemma 1.1].

**Lemma 3.1.10** (Bowditch). *If  $\xi(\Sigma) \geq 2$  then for all  $\alpha, \beta \in \mathcal{S}(\Sigma)$*

$$d_\Sigma(\alpha, \beta) \leq \iota(\alpha, \beta) + 1.$$

There is a logarithmic bound which is better for large distances. See [Hem, Lemma 2.1] for the closed case and [Bow2, Lemma 1.2] for a generalisation to all surfaces of complexity at least 2.

Masur and Minsky observed that the curve graph has infinite diameter whenever  $\xi(\Sigma) \geq 1$  and furthermore they proved the following [MasMi1, Theorem 1.1].

**Theorem 3.1.11** (Masur-Minsky).  *$\mathcal{C}(\Sigma)$  is Gromov hyperbolic.*

The hyperbolicity constants found by Masur and Minsky depend on the complexity of the surface  $\Sigma$ . Several authors later improved this result and showed that there exists a uniform hyperbolicity constant [Aou, Bow4, CIRS, HenPW].

Minsky observed the following [Mins3, Lemma 5.14]

**Lemma 3.1.12** (Minsky). *For any surface  $\Sigma$  with  $\xi(\Sigma) \geq 1$  and any  $x, y \in \overline{\mathcal{C}(\Sigma)}$  there exists a geodesic in  $\mathcal{C}(\Sigma)$  connecting  $x$  and  $y$ .*

Note that this generalises Proposition 2.2.7.

## The pants graph

Let  $\Sigma$  be a surface. A maximal set of disjoint curves on  $\Sigma$  is called a *pants decomposition*. Observe that a pants decomposition consists of exactly  $\xi(\Sigma)$  curves. Cutting the surface along representatives of the curves of a pants decomposition we get  $\chi(\Sigma)$  connected components, each of which is a pair of pants.

Similar to the curve graph whose vertices correspond to curves, we will define a graph whose vertices correspond to pants decompositions. Let  $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}(\Sigma)$  be two distinct pants decompositions of  $\Sigma$ . We say that  $\mathfrak{p}$  and  $\mathfrak{q}$  differ by a *pants move* if the following two conditions hold.

1. There are curves  $\alpha \in \mathfrak{p}$  and  $\beta \in \mathfrak{q}$  such that  $\mathfrak{p} - \{\alpha\} = \mathfrak{q} - \{\beta\}$ .
2. Let  $W$  be the component of  $\Sigma - (\mathfrak{p} - \{\alpha\})$  with  $\xi(W) = 1$ . Then  $\alpha$  and  $\beta$  are curves on  $W$  that are connected by an edge in  $\mathcal{C}(W)$ .

Figure 3.3 gives an example of two pants decompositions on  $\Sigma_{0,5}$  that differ by a pants move.

**Definition 3.1.13** (Pants graph). The *pants graph* of  $\Sigma$  is the graph  $\mathcal{P}(\Sigma)$  with a vertex for every pants decomposition of  $\Sigma$ . Two pants decompositions  $\mathfrak{p}$  and  $\mathfrak{q}$  are connected by an edge in  $\mathcal{P}(\Sigma)$  if and only they differ by a pants move.

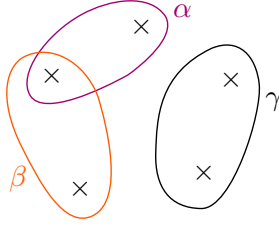


Figure 3.3: The pants decompositions  $\{\alpha, \gamma\}$  and  $\{\beta, \gamma\}$  on  $\Sigma_{0,5}$  differ by a pants move.

This defines a connected graph [HatT, HatLS]. Write  $d_{\mathcal{P}}$  to denote the graph metric on  $\mathcal{P}(\Sigma)$ . If  $\xi(\Sigma) \leq 1$  then a pants decomposition on  $\Sigma$  is the same as a curve on  $\Sigma$ , hence  $\mathcal{P}(\Sigma) \cong \mathcal{C}(\Sigma)$ . This is no longer true when  $\xi(\Sigma) \geq 2$ . Every mapping class induces an automorphism of the pants graph. This defines an action of  $\text{MCG}(\Sigma)$  on  $\mathcal{P}(\Sigma)$ .

Write  $\mathbb{P}(S)$  for the power set of a set  $S$ . Note that any pants decomposition is a set of diameter 1 in the curve graph. There is a map

$$\pi_{\Sigma}: \mathcal{P}(\Sigma) \rightarrow \mathbb{P}(\mathcal{S}(\Sigma))$$

that maps a pants decomposition  $\mathbf{p}$  to the set  $\mathbf{p} \subset \mathcal{S}(\Sigma)$ . Write  $d_{\Sigma}(\mathbf{p}, \mathbf{q})$  for the diameter of the set  $\pi_{\Sigma}(\mathbf{p}) \cup \pi_{\Sigma}(\mathbf{q}) \subset \mathcal{C}(\Sigma)$ . The following lemma shows that  $\pi_{\Sigma}$  is coarsely Lipschitz.

**Lemma 3.1.14.** *Any  $\mathbf{p}, \mathbf{q} \in \mathcal{P}(\Sigma)$  satisfy*

$$d_{\Sigma}(\mathbf{p}, \mathbf{q}) \leq 2d_{\mathcal{P}}(\mathbf{p}, \mathbf{q}) + 1.$$

*Proof.* If  $d_{\mathcal{P}}(\mathbf{p}, \mathbf{q}) = 0$  then  $d_{\Sigma}(\mathbf{p}, \mathbf{q}) = 1$ . Suppose that  $d_{\mathcal{P}}(\mathbf{p}, \mathbf{q}) = 1$ , in other words there is a pants move relating  $\mathbf{p}$  and  $\mathbf{q}$ . The set  $\mathbf{p} \cup \mathbf{q}$  in  $\mathcal{C}(\Sigma)$  has diameter 2. If  $d_{\mathcal{P}}(\mathbf{p}, \mathbf{q}) = k$  for some  $k \geq 1$ , we take a sequence of pants decompositions  $\mathbf{p} = \mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k = \mathbf{q}$  such that  $d_{\mathcal{P}}(\mathbf{p}_i, \mathbf{p}_{i+1}) = 1$  and apply the triangle inequality to find  $d_{\Sigma}(\mathbf{p}, \mathbf{q}) \leq 2k$ .  $\square$

Since  $\mathcal{C}(\Sigma)$  has infinite diameter when  $\xi(\Sigma) \geq 1$ , it follows from the lemma that the pants graph does as well. For a surface  $\Sigma$  with  $\xi(\Sigma) = 2$  and  $\alpha \in \mathcal{S}(\Sigma)$  the full subgraph of  $\mathcal{P}(\Sigma)$  on the vertex set  $\{x \in \mathcal{P}(\Sigma) : \alpha \in x\}$  is a Farey graph. Brock and Farb used these Farey subgraphs to prove the next result [Brocf, Theorems 1.1 and 1.5].

**Theorem 3.1.15** (Brock-Farb).  *$\mathcal{P}(\Sigma)$  is hyperbolic if and only if  $\xi(\Sigma) \leq 2$ .*

Since the pants graph of a surface of complexity 1 is the same as its curve graph, also the Gromov boundaries agree. We will study the Gromov boundary of the pants graph of  $\Sigma_{0,5}$ . We regularly write  $\partial\mathcal{P}$  to denote this boundary, omitting the surface from the notation.

## 3.2 Laminations

This section presents a brief account on laminations. The theory of laminations was introduced by Thurston [Thu1]. Standard references on this topic are Casson and Bleiler's book [CasB] and Bonahon's notes [Bon2]. Throughout the section let  $\Sigma = \Sigma_{g,n}^b$  be a surface with a fixed hyperbolic structure.

### Definitions and properties

A *geodesic* on  $\Sigma$  is the image of a complete geodesic on  $\tilde{\Sigma} \cong \mathbb{H}^2$ . A *lamination* is a non-empty compact subset  $\lambda \subset \Sigma$  that is a disjoint union of simple geodesics. Geodesics contained in  $\lambda$  are called *leaves*. A leaf  $\ell$  of a lamination  $\lambda$  is *isolated* if for every  $x \in \ell$  there is a neighbourhood  $U$  of  $x$  such that  $(U, U \cap \lambda)$  is homeomorphic to  $(\mathbb{I}^2, \{\frac{1}{2}\} \times \mathbb{I})$ . Note that the complement in  $\lambda$  of the isolated leaves is again a lamination.

Let  $\lambda$  be a lamination. For every  $x \in \lambda$  there exists a compact set  $K \subset \mathbb{I}$  with the following property: there is a neighbourhood  $U$  of  $x$  such that  $(U, U \cap \lambda)$  and  $(\mathbb{I}^2, K \times \mathbb{I})$  are homeomorphic [Bon2, Proposition 1]. If  $\lambda$  has no isolated leaves, then for every  $x \in \lambda$  the set  $K$  is a Cantor set [Bon2, Proposition 7].

*Example 3.2.1.* Every curve on  $\Sigma$  has a unique geodesic representative [FarM, Propositions 1.3, 1.6 and 1.10]. Hence we can regard every  $\alpha \in \mathcal{C}(\Sigma)$  as a lamination.

Write  $\mathcal{L}(\Sigma)$  for the set of all laminations on  $\Sigma$ . The *Möbius band at infinity* is the set of unordered pairs of distinct points in  $\partial\mathbb{H}^2$ , topologised as the quotient

$$M_\infty = (\mathbb{S}^1 \times \mathbb{S}^1 - \{(s, s) : s \in \mathbb{S}^1\}) / \sim,$$

where  $(s, t) \sim (t, s)$  for all  $s, t \in \mathbb{S}^1$ . Let  $\tilde{\lambda}$  be the full preimage of  $\lambda$  in  $\mathbb{H}^2$ . The set of pairs of endpoints of geodesics contained in  $\tilde{\lambda}$  is a closed  $\pi_1(\Sigma)$ -invariant subset  $E(\lambda) \subset M_\infty$  which is *unlinked*, that is, any  $x, y \in E(\lambda)$  determine disjoint geodesics in  $\mathbb{H}^2$ . In fact, there is a bijective correspondence between  $\mathcal{L}(\Sigma)$  and the set of closed unlinked  $\pi_1(\Sigma)$ -invariant subsets of  $M_\infty$ . We use this bijection to describe the action of  $\text{MCG}(\Sigma)$  on  $\mathcal{L}(\Sigma)$ . Every homeomorphism  $f : \Sigma \rightarrow \Sigma$  lifts to a homeomorphism of  $\mathbb{H}^2$  that extends to a homeomorphism between boundary circles. This determines

a homeomorphism  $f_\infty: M_\infty \rightarrow M_\infty$  only depending on the isotopy class of  $f$ , and passing through the bijection we obtain an action of  $\text{MCG}(\Sigma)$  on  $\mathcal{L}(\Sigma)$ . For reference see [Thu1, §8.5].

The Hausdorff topology gives  $\mathcal{L}(\Sigma)$  the structure of a compact topological space  $(\mathcal{L}(\Sigma), \mathbf{T}_H)$ . The action of the mapping class group on  $\mathcal{L}(\Sigma)$  described above is by homeomorphisms of the Hausdorff topology. The topological space  $(\mathcal{L}(\Sigma), \mathbf{T}_H)$  does not depend on the choice of hyperbolic structure on  $\Sigma$  [CasB, p.44].

*Remark 3.2.2.* Every puncture of  $\Sigma$  has a neighbourhood that is not entered by any geodesic on  $\Sigma$ . More precisely, there exists a compact subsurface  $\Sigma' \cong \Sigma_g^{b+n}$  of  $\Sigma$  such that any geodesic that intersects  $\Sigma - \Sigma'$  has a self-intersection or does not have compact closure. As a consequence every lamination on  $\Sigma$  is contained in  $\Sigma'$ . Therefore, for studying laminations (or geodesic arcs with endpoints on  $\partial\Sigma$ ) it suffices to study compact surfaces.

*Remark 3.2.3.* The *projective tangent bundle*  $\text{PT}\Sigma$  is the quotient of the tangent bundle of  $\Sigma$  by the equivalence relation  $(x, v) \sim (x, tv)$ , where  $v$  is a non-zero tangent vector at  $x \in \Sigma$  and  $t \in \mathbb{R}_{\neq 0}$ . In particular, the fibre over any  $x \in \Sigma$  is a circle and corresponds to the set of geodesics through  $x$ . Note that  $\text{PT}\Sigma$  is metrisable and every lamination  $\lambda$  on  $\Sigma$  lifts to a compact subset  $\lambda^{\text{PT}}$  of  $\text{PT}\Sigma$ . The Hausdorff topology on  $\mathcal{L}(\Sigma)$  where laminations are seen as subsets of  $\Sigma$  is equivalent to the Hausdorff topology on  $\mathcal{L}(\Sigma)$  where laminations are considered subsets of  $\text{PT}\Sigma$  [CasB, Lemma 3.5].

We are interested in a second topology on  $\mathcal{L}(\Sigma)$ . In the literature this topology is known by the names ‘the geometric topology’ [Thu1, §8.10], ‘the coarse Hausdorff topology’ [Ham, §1] and ‘the topology of superconvergence’ [Gab1, §3]. We choose to follow Gabai’s nomenclature.

**Definition 3.2.4** (Topology of superconvergence). Let  $(\lambda_i)_{i \in \mathbb{N}}$  and  $\lambda$  be laminations on  $\Sigma$ . The sequence  $(\lambda_i)_{i \in \mathbb{N}}$  *superconverges* to  $\lambda$  if for every  $(x, v) \in \lambda^{\text{PT}}$  there exist  $(x_i, v_i) \in \lambda_i^{\text{PT}}$  so that  $(x_i, v_i)_{i \in \mathbb{N}}$  converges to  $(x, v)$  in  $\text{PT}\Sigma$ . Let  $(\mathcal{L}(\Sigma), \mathbf{T}_s)$  be the topological space with the following closed sets: a set  $C \subset \mathcal{L}(\Sigma)$  is closed if and only if for every sequence in  $C$  that superconverges to  $\lambda \in \mathcal{L}(\Sigma)$  we have that  $\lambda \in C$ . This topology is the *topology of superconvergence* on  $\mathcal{L}(\Sigma)$ .

On a subset  $X \subset \mathcal{L}(\Sigma)$  let  $(X, \mathbf{T}_s)$  be the subspace topology of  $(\mathcal{L}(\Sigma), \mathbf{T}_s)$ , called the *topology of superconvergence* on  $X$ .

*Remark 3.2.5.* Superconvergence defines a ‘limit operator’  $L$  on  $\mathcal{L}(\Sigma)$  and the pair  $(\mathcal{L}(\Sigma), L)$  is an ‘ $\mathcal{S}^*$ -space’. The topology of superconvergence  $\mathbf{T}_s$  on  $\mathcal{L}(\Sigma)$  is the ‘Fréchet topology’ induced by  $L$ . See Appendix A for the definitions of limit operator,

$\mathcal{S}^*$ -space and Fréchet topology. By Lemma A.7.(iii) the topology of superconvergence on  $X \subset \mathcal{L}(\Sigma)$  is the same as the Fréchet topology on  $X$  induced by the restriction of  $L$ .

*Remark 3.2.6.* The identity map  $(\mathcal{L}(\Sigma), \mathbf{T}_H) \rightarrow (\mathcal{L}(\Sigma), \mathbf{T}_s)$  is continuous. This follows from the facts that the Hausdorff topology is metrisable and any Hausdorff convergent sequence of laminations is also superconvergent. It follows that the topology of superconvergence is coarser than the Hausdorff topology. Observe that  $\text{MCG}(\Sigma)$  acts on  $(\mathcal{L}(\Sigma), \mathbf{T}_s)$  by homeomorphisms.

*Remark 3.2.7.*  $(\mathcal{L}(\Sigma), \mathbf{T}_s)$  is not Hausdorff. For instance, if  $\alpha$  and  $\beta$  are disjoint curves, then the lamination  $\alpha \cup \beta$  is contained in every neighbourhood of  $\alpha$  as well as in every neighbourhood of  $\beta$ .

A *domain* in  $\Sigma$  is a closed connected subsurface  $W \subset \Sigma$  such that  $\xi(W) \geq 1$  and every component of  $\partial W - \partial \Sigma$  is an essential and non-peripheral simple closed curve on  $\Sigma$ . A domain is *proper* if it does not equal the full surface. We say that a domain  $W$  has *geodesic boundary* if every component of  $\partial W - \partial \Sigma$  is a simple closed geodesic on  $\Sigma$ . Every domain in a planar surface is isotopic to a domain with geodesic boundary. A *generalised domain* is a closed subsurface  $W \subset \Sigma$  that is a union of domains and annuli whose core curve is essential and non-peripheral in  $\Sigma$ .

Let  $\lambda$  be a lamination on  $\Sigma$ . There exists  $\varepsilon > 0$  such that the  $\varepsilon$ -neighbourhood  $N_\varepsilon(\lambda)$  deformation retracts to  $N_\delta(\lambda)$  for all  $0 < \delta < \varepsilon$ . We call such  $N_\varepsilon(\lambda)$  a *regular neighbourhood* of  $\lambda$ . A generalised domain is the *support* of  $\lambda$ , denoted  $\text{Supp}(\lambda)$ , if it is isotopic to a subsurface  $W \subset \Sigma$  with the following properties:

1.  $\lambda \subset W$ ,
2.  $\partial W \subset \text{fr } N_\varepsilon(\lambda)$ , and
3. every component of  $\text{fr } N_\varepsilon(\lambda) - \partial W$  is peripheral or non-essential. In other words,  $W$  is the smallest domain subject to the first two conditions.

Say that a lamination is *filling* if it is supported on  $\Sigma$ , or equivalently, if it intersects every simple closed geodesic on  $\Sigma$ .

A lamination is *minimal* if its only sublamination is itself. Equivalently,  $\lambda$  is minimal if and only if every leaf of  $\lambda$  is dense in  $\lambda$ . A minimal lamination that is not a simple closed geodesic has uncountably many leaves [Lev, p.121] and fills a domain. Any lamination is a finite union of minimal laminations and isolated leaves [Bon2, Proposition 3].

**Definition 3.2.8** (Ending lamination). A lamination is *ending* if it is minimal and filling. Write  $\mathcal{EL}(\Sigma)$  for the set of ending laminations on  $\Sigma$ .

*Remark 3.2.9.* The notion of ending laminations was first developed by Thurston [Thu1, Thu2]. He conjectured that every hyperbolic 3-manifold  $M$  with finitely generated fundamental group is uniquely determined by its topology and its ‘end invariants’. The fact that every end of  $M$  has an associated end invariant is a consequence of (the proof of) the ‘Tameness Theorem’ [Bon1, Ago, CalG]. In certain cases these end invariants are ending laminations, hence the name. For a detailed exposition see, for instance, [Mins1, Bow3]. This conjecture – now known as the ‘Ending Lamination Theorem’ – was proven in the ‘indecomposable’ case by Minsky, Brock and Canary, who also announced a proof in the general case [Mins3, BrocCM]. Alternative approaches were proposed in [Ree, Bow1, Som].

Gabai studied the connectivity of  $\mathcal{EL}(\Sigma)$  [Gab1, Theorem 0.1].

**Theorem 3.2.10** (Gabai).  $(\mathcal{EL}(\Sigma), \mathbf{T}_s)$  is path connected and locally path connected.

Turning to the specific case of the five-times punctured sphere, we consider the following type of lamination, following [BrocM].

**Definition 3.2.11** (Boundary lamination). A lamination on  $\Sigma_{0,5}$  is a *boundary lamination* if it is minimal and has more than one leaf. The set of boundary laminations is denoted  $\mathcal{BL}$ .

Clearly ending laminations on  $\Sigma_{0,5}$  are also boundary laminations. Let  $W_\alpha$  denote the domain in  $\Sigma_{0,5}$  with geodesic boundary  $\alpha \in \mathcal{S}(\Sigma_{0,5})$ . A boundary lamination that is not ending is supported on  $W_\alpha$  for some  $\alpha \in \mathcal{S}(\Sigma_{0,5})$ . It follows that we can write  $\mathcal{BL}$  as a union of disjoint sets

$$\mathcal{BL} = \mathcal{EL}(\Sigma_{0,5}) \cup \bigcup_{\alpha \in \mathcal{S}(\Sigma_{0,5})} \mathcal{EL}(W_\alpha). \quad (3.1)$$

**Definition 3.2.12** (Almost boundary lamination). A lamination  $\lambda$  on  $\Sigma_{0,5}$  is an *almost boundary lamination* if it is a boundary lamination or it is the union of a boundary lamination and a simple closed geodesic. Denote by  $\lambda^*$  the sublamination of  $\lambda$  that is a boundary lamination. Write  $\mathcal{ABL}$  for the set of almost boundary laminations.

*Remark 3.2.13.* Almost boundary laminations are the ‘almost minimal almost filling laminations’ on the five-times punctured sphere, defined by Gabai [Gab1, Definition 1.2].

## Measured laminations

In this subsection we will endow laminations with transverse measures. Roughly, a transverse measure is a map that assigns a length to every transverse arc and this length is invariant under ‘sliding’ the arc along the lamination. We will make this precise below. For the description of  $\mathcal{ML}(\Sigma)$  and its topology we follow the discussion presented in [PeH, §1.7].

Only for the purpose of this subsection, all arcs on  $\Sigma$  are required to be differentiable but not necessarily proper. For a lamination  $\lambda \in \mathcal{L}(\Sigma)$ , let  $\mathbb{A}(\lambda)$  denote the set of all arcs on  $\Sigma$  that are transverse to  $\lambda$  and whose endpoints lie in  $\Sigma - \lambda$ .

**Definition 3.2.14** (Measured lamination). Let  $\lambda$  be a lamination on  $\Sigma$  or the empty set. A *transverse measure* on  $\lambda$  is a function

$$m: \mathbb{A}(\lambda) \rightarrow \mathbb{R}_{\geq 0}$$

satisfying

1.  $m(\mathbf{a}) = m(\mathbf{b})$  whenever  $\mathbf{a}$  is isotopic to  $\mathbf{b}$  through elements of  $\mathbb{A}(\lambda)$ ;
2. if  $\mathbf{a} \in \mathbb{A}(\lambda)$  decomposes as a union of at most countably many arcs  $\mathbf{a}_i \in \mathbb{A}(\lambda)$  that intersect only in their endpoints, then

$$m(\mathbf{a}) = \sum_i m(\mathbf{a}_i);$$

3. for any  $\mathbf{a} \in \mathbb{A}(\lambda)$ ,  $m(\mathbf{a}) > 0$  if and only if  $\mathbf{a} \cap \lambda \neq \emptyset$ .

In this case, the pair  $(\lambda, m)$  is called a *measured lamination*.

*Example 3.2.15.* A basic example of a measured lamination is a single simple closed geodesic with counting measure. Let  $\lambda$  be a simple closed geodesic on  $\Sigma$  and for any transverse arc  $\mathbf{a}$ , define  $m(\mathbf{a})$  to be the number of crossings of  $\lambda$  and  $\mathbf{a}$ .

*Example 3.2.16.* If  $m$  is a transverse measure on  $\lambda$  then also  $rm$  is, for every positive  $r \in \mathbb{R}$ . If  $(\lambda, m)$  and  $(\lambda', m')$  are measured laminations such that  $\lambda$  and  $\lambda'$  are disjoint, then  $(\lambda \cup \lambda', m + m')$  is again a measured lamination.

Let  $\mathcal{ML}(\Sigma)$  denote the set of measured laminations on  $\Sigma$ . To carefully describe the topology on this set we need some terminology. Consider a measured lamination  $(\lambda, m)$ . Recall that  $M_\infty$  denotes the Möbius band at infinity (page 23). Let  $\tilde{\lambda}$  be the full preimage of  $\lambda$  in  $\mathbb{H}^2$  and let  $E(\lambda) \subset M_\infty$  be the set of pairs of endpoints of geodesics in  $\tilde{\lambda}$ . The transverse measure  $m$  determines a  $\pi_1(\Sigma)$ -invariant measure



$m_\infty$  on  $M_\infty$  as follows. For any point  $x \in \tilde{\lambda}$  there is a neighbourhood  $U$  of  $x$  in  $\mathbb{H}^2$ , a compact set  $K \subset \text{int}(I)$  and a homeomorphism  $f: U \rightarrow I^2$  so that  $f(\tilde{\lambda} \cap U) = K \times I$ . The endpoints of the leaves of  $\tilde{\lambda}$  that pass through  $U$  determine a closed subset  $E(\lambda, U)$  of  $M_\infty$ . If  $V \subset M_\infty$  is an open set so that  $V \cap E(\lambda) = V \cap E(\lambda, U)$ , define

$$m_\infty(V) = m(\mathbf{a})$$

where  $\mathbf{a} \subset U$  is the arc  $f^{-1}(I \times \{1\})$ . This uniquely determines a measure on  $M_\infty$ , and the assignment  $(\lambda, m) \mapsto m_\infty$  is an injection of  $\mathcal{ML}(\Sigma)$  into the space of measures supported on  $M_\infty$  [PeH, Proposition 1.7.1].

**Definition 3.2.17** (Measure topology). The topology on  $\mathcal{ML}(\Sigma)$  is the subspace topology of the weak topology on the measures supported on  $M_\infty$ .

An equivalent but perhaps more intuitive description of the topology on  $\mathcal{ML}(\Sigma)$  can be given using generic arcs. We encountered this characterisation in [Wic, p.5]. Say that a differentiable arc  $\mathbf{a}$  in  $\Sigma$  is *generic* if it is transverse to every simple geodesic on  $\Sigma$ . Every differentiable arc can be approximated by generic arcs [Bon2, p.19]. A sequence of measured geodesic laminations  $((\lambda_i, m_i))_{i \in \mathbb{N}}$  converges to  $(\lambda, m)$  if and only if for every generic arc  $\mathbf{a} \in \mathbb{A}(\lambda)$ , there exists some  $N \in \mathbb{N}$  such that  $\mathbf{a} \in \mathbb{A}(\lambda_i)$  for all  $i > N$ , and the values  $m_i(\mathbf{a})$  converge to  $m(\mathbf{a})$ .

Not every lamination admits a transverse measure – a lamination that does is called *measurable*. Example 3.2.15 shows that simple closed geodesics are measurable. More generally, every minimal lamination is measurable [Bon2, Proposition 9]. Since any lamination  $\lambda$  is a finite union of minimal laminations and isolated leaves, we obtain a measurable lamination by deleting all isolated bi-infinite leaves of  $\lambda$ . For measurable laminations  $\lambda$  and  $\mu$  we write

$$\iota(\lambda, \mu) = 0$$

if and only if  $\lambda \cup \mu$  is a lamination. This extends the definition of the intersection number of disjoint curves.

There is map from  $\mathcal{ML}(\Sigma)$  to  $\mathcal{L}(\Sigma)$  that associates to every measured lamination its underlying lamination. When  $\mathcal{L}(\Sigma)$  is equipped with the Hausdorff topology, this map is not continuous. Thurston observed the following [Thu1, Proposition 8.10.3].

**Proposition 3.2.18** (Thurston). *The measure-forgetting map  $\mathcal{ML}(\Sigma) \rightarrow (\mathcal{L}(\Sigma), \mathbf{T}_s)$ ,  $(\lambda, m) \mapsto \lambda$  is continuous.*

We define an equivalence relation on the set of non-trivial measured laminations. Write  $(\lambda, m) \sim (\lambda', m')$  if and only if  $\lambda = \lambda'$  and  $m = rm'$  for some  $r \in \mathbb{R}_{>0}$ . An equivalence class of measured laminations is a *projective lamination*. Let  $\mathcal{PML}(\Sigma)$  be the set projective laminations. Let

$$\Psi: \mathcal{ML}(\Sigma) - \{(\emptyset, 0)\} \rightarrow \mathcal{PML}(\Sigma)$$

be the *projectivisation map* and endow  $\mathcal{PML}(\Sigma)$  with the quotient topology. It follows from Proposition 3.2.18 and the universal property of the quotient topology that the *measure-forgetting map*

$$\Phi: \mathcal{PML}(\Sigma) \rightarrow \mathcal{L}(\Sigma), \quad [(\lambda, m)] \mapsto \lambda$$

is continuous with respect to the topology of superconvergence on  $\mathcal{L}(\Sigma)$ . The image of  $\Phi$  is denoted  $\mathcal{UML}(\Sigma)$  and is equipped with the quotient topology from  $\mathcal{PML}(\Sigma)$ .

*Example 3.2.19.* Any transverse measure on a simple closed geodesic is a multiple of the counting measure. Consequently, a curve uniquely determines a projective lamination. We will consider  $\mathcal{S}(\Sigma)$  as a subset of  $\mathcal{PML}(\Sigma)$  in this way.

There is another viewpoint on projective laminations that we would like to mention. Let  $\mathbb{R}^{\mathcal{S}(\Sigma)}$  be the space of functions  $\mathcal{S}(\Sigma) \rightarrow \mathbb{R}$  endowed with the topology of pointwise convergence. Write  $P(\mathbb{R}^{\mathcal{S}(\Sigma)})$  for the associated projective space, endowed with the quotient topology so that the projection  $P: \mathbb{R}^{\mathcal{S}(\Sigma)} \rightarrow P(\mathbb{R}^{\mathcal{S}(\Sigma)})$  is continuous. The intersection number of curves induces a map

$$\iota_*: \mathcal{S}(\Sigma) \rightarrow \mathbb{R}^{\mathcal{S}(\Sigma)},$$

where  $\iota_*(\alpha)$  is defined to be the function  $\beta \mapsto \iota(\alpha, \beta)$ . Let  $\mathcal{PS}(\Sigma)$  be the closure of  $P \circ \iota_*(\mathcal{S}(\Sigma))$  in  $P(\mathbb{R}^{\mathcal{S}(\Sigma)})$ . In the 1970s Thurston announced the following celebrated theorems [Thu4, Theorems 1 and 2].

**Theorem 3.2.20** (Thurston). *There is a homeomorphism  $\mathcal{PML}(\Sigma) \rightarrow \mathcal{PS}(\Sigma)$  with  $\alpha \mapsto P \circ \iota_*(\alpha)$  for every  $\alpha \in \mathcal{S}(\Sigma)$ .*

**Theorem 3.2.21** (Thurston).  $\mathcal{PS}(\Sigma) \cong \mathbb{S}^{2\xi(\Sigma)-1}$ .

*Remark 3.2.22.* Since  $P \circ \iota_*(\mathcal{S}(\Sigma))$  is a dense subset of  $\mathcal{PS}(\Sigma)$ , it follows from Theorem 3.2.20 that curves are dense in  $\mathcal{PML}(\Sigma)$ .

We describe the action of  $\text{MCG}(\Sigma)$  on  $\mathcal{PML}(\Sigma)$  using Thurston's identification. Any mapping class defines a bijection  $f: \mathcal{S}(\Sigma) \rightarrow \mathcal{S}(\Sigma)$  which in turn induces

a continuous map of function spaces

$$f_*: \mathbb{R}^{\mathcal{S}(\Sigma)} \rightarrow \mathbb{R}^{\mathcal{S}(\Sigma)}, \quad \varphi \mapsto \varphi \circ f^{-1}.$$

Observe that  $rf_*(\varphi) = f_*(r\varphi)$  for every  $r \in \mathbb{R}$  and  $\varphi: \mathcal{S}(\Sigma) \rightarrow \mathbb{R}$ , so the map

$$f_*^P: P(\mathbb{R}^{\mathcal{S}(\Sigma)}) \rightarrow P(\mathbb{R}^{\mathcal{S}(\Sigma)}), \quad [\varphi] \mapsto [f_*\varphi],$$

is well defined. It follows from the universal property of the quotient topology that  $f_*^P$  is continuous. Note that  $(f^{-1})_*^P$  is a continuous inverse of  $f_*^P$ . The diagram

$$\begin{array}{ccccc} \mathcal{S}(\Sigma) & \xrightarrow{\iota_*} & \mathbb{R}^{\mathcal{S}(\Sigma)} & \xrightarrow{P} & P(\mathbb{R}^{\mathcal{S}(\Sigma)}) \\ \downarrow f & & \downarrow f_* & & \downarrow f_*^P \\ \mathcal{S}(\Sigma) & \xrightarrow{\iota_*} & \mathbb{R}^{\mathcal{S}(\Sigma)} & \xrightarrow{P} & P(\mathbb{R}^{\mathcal{S}(\Sigma)}) \end{array}$$

commutes, hence  $f_*^P$  restricts to a homeomorphism of  $\mathcal{PS}(\Sigma)$ . This defines an action of  $\text{MCG}(\Sigma)$  on  $\mathcal{PS}(\Sigma)$ . On the subset of curves in  $\mathcal{PML}(\Sigma)$  this action agrees with the usual action of  $\text{MCG}(\Sigma)$  on  $\mathcal{S}(\Sigma)$ .

*Example 3.2.23.* Consider a curve  $\alpha \in \mathcal{S}(\Sigma)$  and its Dehn twist  $T_\alpha$ . Let  $\beta$  be any curve that intersects  $\alpha$ . We will show that the sequence  $(T_\alpha^n(\beta))_{n=1}^\infty$  converges to  $\alpha$  in  $\mathcal{PML}(\Sigma)$ . For every  $\gamma \in \mathcal{S}(\Sigma)$  we have

$$|\iota(T_\alpha^n(\beta), \gamma) - |n|\iota(\alpha, \beta)\iota(\alpha, \gamma)| \leq \iota(\beta, \gamma)$$

[FarM, Proposition 3.4], from which it follows that

$$\frac{\iota(T_\alpha^n(\beta), \gamma)}{|n|\iota(\alpha, \beta)} \rightarrow \iota(\alpha, \gamma)$$

as  $n \rightarrow \infty$ . This shows that the sequence  $((|n|\iota(\alpha, \beta))^{-1}\iota_*(T_\alpha^n(\beta)))_n$  converges to  $\iota_*(\alpha)$  in  $\mathbb{R}^{\mathcal{S}(\Sigma)}$ , hence  $(P \circ \iota_*(T_\alpha^n(\beta)))_{n \in \mathbb{N}}$  converges to  $P \circ \iota_*(\alpha)$  in  $P(\mathbb{R}^{\mathcal{S}(\Sigma)})$ .

More generally, if  $[(\lambda, m)]$  is any projective lamination with  $\iota(\lambda, \alpha) \neq 0$  then the sequence  $(T_\alpha^n \cdot [(\lambda, m)])_{n \in \mathbb{N}}$  converges to  $\alpha$  in  $\mathcal{PML}(\Sigma)$ . Curves are dense in  $\mathcal{PML}(\Sigma)$ , so we can find a sequence  $(\beta_i)$  of curves that converges to  $[(\lambda, m)]$ . The measure-forgetting map  $\Phi$  is continuous and  $\iota(\lambda, \alpha) \neq 0$ , from which it follows that  $\iota(\beta_i, \alpha) > 0$  whenever  $i$  is large enough. Now  $T_\alpha^n(\beta_i) \rightarrow T_\alpha^n \cdot [(\lambda, m)]$  as  $i \rightarrow \infty$  and  $T_\alpha^n(\beta_i) \rightarrow \alpha$  as  $n \rightarrow \infty$ . A diagonal argument shows that  $T_\alpha^n \cdot [(\lambda, m)] \rightarrow \alpha$  in  $\mathcal{PML}(\Sigma)$ .

If  $\alpha \in \mathcal{S}(\Sigma)$  bounds a twice punctured disc in  $\Sigma$  then  $H_\alpha^2 = T_\alpha$ . Hence for

any projective lamination  $[(\lambda, m)]$  with  $\iota(\lambda, \alpha) \neq 0$  the sequence  $(H_\alpha^n \cdot [(\lambda, m)])_{n \in \mathbb{N}}$  converges to  $\alpha$  in  $\mathcal{PML}(\Sigma)$ .

Observe that  $\mathcal{EL}(\Sigma)$  inherits a topology from  $\mathcal{ML}(\Sigma)$  by taking the subspace topology of  $\mathcal{UML}(\Sigma)$ . This topology is equivalent to the topology of superconvergence [Ham, p.3]. Klarreich proved the following theorem [Kla].

**Theorem 3.2.24** (Klarreich). *Let  $W \subset \Sigma$  be a domain.*

- (i) *There is a  $\text{MCG}(W)$ -equivariant homeomorphism  $\Theta_W: \partial\mathcal{C}(W) \rightarrow (\mathcal{EL}(W), \mathbf{T}_s)$ .*
- (ii) *A sequence  $(\alpha_i)_{i \in \mathbb{N}}$  in  $\mathcal{C}(W)$  converges to  $x \in \partial\mathcal{C}(W)$  if and only if  $(\alpha_i)_{i \in \mathbb{N}}$  superconverges to  $\Theta_W(x)$ .*

This formulation of Klarreich's theorem and an alternative proof were obtained by Hamenstädt in [Ham, Theorem 1.1]. Recently, Pho-On published a direct elementary proof [Pho].

**Definition 3.2.25** (Projective (almost) boundary lamination). *A projective boundary lamination is a projective lamination on  $\Sigma_{0,5}$  whose underlying lamination is a boundary lamination. Let  $\mathcal{PBL} = \Phi^{-1}(\mathcal{BL})$  denote the set of projective boundary laminations. A projective almost boundary lamination is a projective lamination on  $\Sigma_{0,5}$  whose underlying lamination is an almost boundary lamination. The set of projective almost boundary laminations is denoted  $\mathcal{PABL} = \Phi^{-1}(\mathcal{ABL})$ . Both  $\mathcal{PBL}$  and  $\mathcal{PABL}$  are equipped with the subspace topology of  $\mathcal{PML}(\Sigma_{0,5})$ .*

### 3.3 Train tracks

Thurston introduced 'train tracks' to endow  $\mathcal{PML}(\Sigma)$  with local coordinates [Thu1]. These coordinates help to translate abstract questions about lamination spaces into concrete linear algebra. In this section we review some of Thurston's theory of train tracks. More details can be found in Penner and Harer's book [PeH] and Mosher's notes [Mos]. Results that are stated without proof are taken directly from Penner and Harer's book (and are referenced accordingly).

#### Definitions and properties

Let  $\Sigma = \Sigma_{g,n}$  be a punctured surface without boundary. Assume that  $\Sigma$  is equipped with a smooth structure. In fact, there exists a unique smooth structure on  $\Sigma$ . For a proof of this classical result see, for instance, [Hat2].

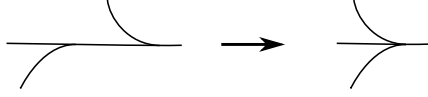


Figure 3.4: A shift.

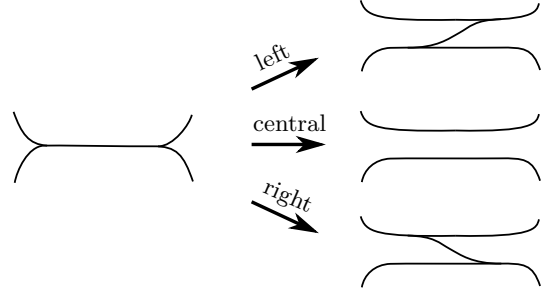


Figure 3.5: Left, central and right splits.

**Definition 3.3.1** (Pretrack). A *pretrack* on  $\Sigma$  is an embedded finite graph  $\tau \subset \Sigma$  that satisfies the following conditions. Its edges (called *branches*) are  $C^1$ -arcs with well-defined tangent vectors at the endpoints. For any vertex (called *switch*)  $s$  of  $\tau$  there is a unique tangent line  $L \in \text{PT}_s \Sigma$  such that any edge incident to  $v$  is tangent to  $L$  at  $v$ . Through each switch there is a  $C^1$ -path which is embedded in  $\tau$  and which contains the switch in its interior. Each component of  $\tau$  that is a curve has a unique bivalent switch. All other switches have valence at least 3.

By convention branches are open 1-cells, so that a pretrack is a disjoint union of its switches and branches. A *half-branch* of a pretrack  $\tau$  is an edge in the barycentric subdivision of  $\tau$ . Assume that half-branches are directed towards the incident switch  $s$ , so that their tangent line at  $s$  is oriented. The half-branches incident to a switch  $s$  of valence  $n$  are partitioned into two sets

$$\{\{h_1, \dots, h_k\}, \{h_{k+1}, \dots, h_n\}\},$$

where the tangent of the half-branch  $h_i$  at  $s$  is a positive multiple of  $v$  if and only if  $1 \leq i \leq k$ . Here  $v$  is a unit tangent vector at  $s$  that spans the tangent line  $L \in \text{PT}_s \Sigma$ . Note that choosing  $-v$  instead of  $v$  does not affect the partition.

A pretrack  $\tau$  is *semi-generic* if the partition of the half-branches incident to any switch contains a singleton. A semi-generic pretrack is *generic* if every switch has valence at most 3. Suppose that  $\tau$  is a semi-generic pretrack. Let  $s$  be a switch of  $\tau$  whose incident half-branches are partitioned  $\{\{h_1\}, \{h_2, h_3, \dots, h_n\}\}$  for some  $n \geq 3$ . We say that  $h_1$  is a *large* half-branch and  $h_2, \dots, h_n$  are *small* half-branches. A branch that consists of two large half-branches is called *large*. A *small* branch consists of two small half-branches and a *mixed* branch consist of a large and a small half-branch.

A *shift* is an operation on a semi-generic pretrack that contracts a mixed branch to a point, merging two switches. The inverse operation of a shift is called a

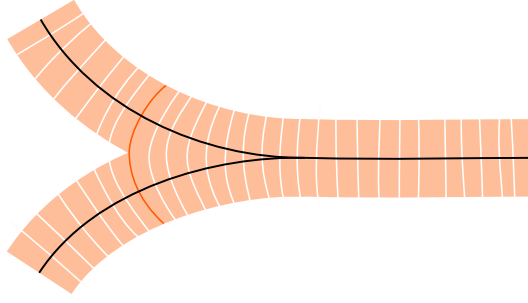


Figure 3.6: A local picture of a tie neighbourhood near a switch.

shift as well. See Figure 3.4. A *Left*, *right* or *central split* is an operation on a large branch of a generic pretrack, depicted in Figure 3.5. The inverse operation of a split is called a *fold*.

*Remark 3.3.2.* Semi-generic pretracks are considered up to smooth isotopy and shifting, usually without explicit mention.

**Definition 3.3.3** (Tie neighbourhood). Let  $\tau$  be a generic pretrack. A regular neighbourhood  $N = N(\tau)$  is called a *tie neighbourhood* of  $\tau$  if the boundary of  $N$  is piecewise smooth and  $N$  is foliated by smooth arcs transverse to  $\tau$ , called *ties*. A tie  $t$  is *regular* if it has a neighbourhood  $U \subset N$  so that there is a homeomorphism  $(U, t) \cong (I \times I, \{\frac{1}{2}\} \times I)$  that takes the foliation of  $U$  to the vertical foliation of  $I \times I$ . Non-regular ties are called *singular*.

Figure 3.6 gives a local picture of a tie neighbourhood near a switch. Given a tie neighbourhood  $N = N(\tau)$ , there exists a *tie-collapsing map*

$$F: (\Sigma, N, \Sigma - N) \rightarrow (\Sigma, \tau, \Sigma - \tau)$$

satisfying the conditions:

- $F: \Sigma \rightarrow \Sigma$  is homotopic to the identity.
- $F: \Sigma - N \rightarrow \Sigma - \tau$  is a homeomorphism.
- For every  $x \in \tau$ ,  $F^{-1}(x)$  is a tie of  $N$ , called the *tie over  $x$* . The tie over  $x \in \tau$  is singular if and only if  $x$  is a switch of  $\tau$ .

Note that the tie over  $x \in \tau$  is in general not the same as the tie that intersects  $x$ , when  $\tau$  is seen as a subset of  $N(\tau)$ .

**Definition 3.3.4** (Train path). Let  $\tau$  be a pretrack. For a connected set  $R \subset \mathbb{R}$  a *train path* on  $\tau$  is a  $C^1$ -immersion  $p: R \rightarrow \Sigma$  with the properties that  $p(R) \subset \tau$  and  $p(k)$  is a switch of  $\tau$  if and only if  $k \in \mathbb{Z} \cap R$ . A train path is *closed* if it is periodic.

We consider train paths  $p: R \rightarrow \Sigma$  and  $p': R' \rightarrow \Sigma$  on  $\tau$  to be the same if  $p = p' \circ r$  for some function  $r: R \rightarrow R'$  that restricts to a bijection  $R \cap \mathbb{Z} \rightarrow R' \cap \mathbb{Z}$ .

**Definition 3.3.5** (Carrying). Let  $\lambda$  be a lamination or a pretrack and let  $\tau$  be a pretrack. We say that  $\lambda$  is *carried by*  $\tau$  (denoted  $\lambda \prec \tau$ ) if for every tie neighbourhood  $N(\tau)$  there exists a homeomorphism  $\varphi: \Sigma \rightarrow \Sigma$  satisfying the following conditions:  $\varphi$  is a  $C^1$ -immersion,  $\varphi$  is isotopic to the identity and  $\varphi(\lambda) \subset \text{int } N(\tau)$ . The map  $\varphi$  is called a *carrying map*. Say that  $\lambda$  *traverses* a branch  $b$  of  $\tau$  if  $b \subset F\varphi(\lambda)$ , where  $F$  is a tie-collapsing map of  $N(\tau)$ . If  $\lambda$  traverses every branch of  $\tau$ ,  $\lambda$  is *fully carried* by  $\tau$ .

Note that if  $\tau$  and  $\sigma$  are semi-generic pretracks and  $\sigma$  is isotopic to  $\tau$  or arises from  $\tau$  by a shift or a split, then  $\sigma \prec \tau$ . It is not hard to see that carrying is transitive. Suppose that  $\lambda \prec \tau$  and  $\tau \prec \sigma$ . Take any tie neighbourhood  $N(\sigma)$  and let  $\varphi$  be a carrying map so that  $\varphi(\tau) \subset \text{int } N(\sigma)$ . Since  $\tau$  is compact, there exists an open set  $U$  so that  $\varphi(\tau) \subset U \subset N(\sigma)$ . Take a tie neighbourhood  $N(\tau) \subset \varphi^{-1}(U)$  and a carrying map  $\psi$  so that  $\psi(\lambda) \subset N(\tau)$ . The composition  $\varphi\psi$  is again a carrying map, showing that  $\lambda \prec \sigma$ .

To upgrade the definition of a pretrack to that of a train track we want to put some restrictions on the types of complementary regions. A *surface with corners* is a surface  $K$  equipped with an atlas of charts mapping to local models, so that the transition functions are  $C^1$ . For every  $z \in K$  there is a set  $X \subset \mathbb{R}^2$ , an open set  $U \subset K$  containing  $z$  and a chart  $f: U \rightarrow X \subset \mathbb{R}^2$  so that  $f(z)$  is the origin and  $f(U)$  is an open set in  $X$ . The set  $X$  is called the *local model* at  $z$  and is only allowed to be one of the following four types. If  $X = \mathbb{R}^2$  we say that  $z$  is an *interior point*; if  $X$  is a closed half-plane then  $z$  is a *regular boundary point*; if  $X$  is a closed quadrant then  $z$  is a *corner*; lastly,  $p$  is called a *cusp* if

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, f(x) \leq y \leq g(x)\},$$

where  $f, g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  are two  $C^1$ -functions which vanish to first order at  $x = 0$  and which satisfy  $f(x) < g(x)$  for all  $x > 0$ . When  $K$  has no corners, we say it is a *surface with cusps*.

For surfaces with corners we generalise the Euler characteristic as follows.

**Definition 3.3.6** (Euler index). The *Euler index* of a surface with corners  $K$  is

defined as

$$\text{index}(K) = \chi(K) - \frac{1}{2}|\{\text{cusps of } K\}| - \frac{1}{4}|\{\text{corners of } K\}|.$$

The Euler index is additive under gluing in the following sense: if  $K, K' \subset \Sigma$  are surfaces with cusps whose interiors are disjoint and  $K \cup K'$  is again a surface with cusps, then  $\text{index}(K \cup K') = \text{index}(K) + \text{index}(K')$ . The closure of a complementary region of a pretrack is a surface with cusps.

**Definition 3.3.7** (Train track). A pretrack  $\tau$  is a *train track* if the Euler index of the closure of every component of  $\Sigma - \tau$  is negative. A train track is *maximal* if every closed complementary region has index  $-\frac{1}{2}$ .

Every train track is a subtrack of a maximal train track. If  $\tau, \sigma$  are train tracks and there exists a mapping class  $f$  with  $f\tau = \sigma$ , we say that  $\tau$  and  $\sigma$  have the same *combinatorial type*. Note that on every surface there are only finitely many combinatorial types of train tracks.

A surface with corners that is homeomorphic to a disc and has a total of  $n$  corners and cusps is called an *n-gon*. In particular, when  $n = 1, 2$  or  $3$  we say it is a *monogon*, *bigon* or *trigon*, respectively. Note that a train track  $\tau$  is maximal if and only if every complementary region is a trigon or a once punctured monogon.

It is not hard to see that, if two semi-generic train tracks  $\sigma$  and  $\tau$  differ by isotopy and shifts only, then  $\sigma \prec \tau$  and  $\tau \prec \sigma$ . If  $\sigma$  is obtained from  $\tau$  by performing a split, then  $\sigma \prec \tau$ . The following proposition gives a converse to this statement [Mos, Proposition 3.14.1]

**Proposition 3.3.8.** *Suppose that  $\sigma$  and  $\tau$  are semi-generic train tracks and  $\sigma$  is fully carried by  $\tau$ . Then there exist a finite sequence of semi-generic train tracks*

$$\sigma = \sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_n = \tau$$

*such that, up to isotopy and shifting, every  $\sigma_i$  is obtained from  $\sigma_{i-1}$  by performing a split.*

Let  $g$  be a  $C^1$ -embedded curve on  $\Sigma$  and let  $\tau$  be a train track. If  $g$  is transverse to  $\tau$  and no component of  $\Sigma - \tau - g$  is a bigon, we say that  $g$  *hits  $\tau$  efficiently*.

**Definition 3.3.9** (Recurrence). A train track  $\tau$  is *recurrent* if for every branch  $b$  of  $\tau$  there exist a curve carried by  $\tau$  that traverses  $b$ . It is *transversely recurrent* if for every branch  $b$  of  $\tau$  there exists an embedded curve that hits  $\tau$  efficiently and



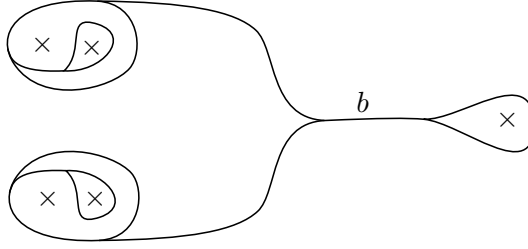


Figure 3.7: A maximal recurrent train track on  $\Sigma_{0,5}$  that is not transversely recurrent. No simple closed curve that intersects the branch  $b$  hits the train track efficiently.

intersects  $b$  at least once. If  $\tau$  is both recurrent and transversely recurrent it is *birecurrent*. A maximal birecurrent train track is called *complete*.

If  $\tau$  is a transversely recurrent train track and  $\sigma \prec \tau$ , then  $\sigma$  is also transversely recurrent [PeH, Lemma 1.3.3].

*Remark 3.3.10.* On the four-times punctured sphere every train track is transversely recurrent. To see this, note that every complementary component of a maximal train track  $\tau$  on  $\Sigma_{0,4}$  is a once-punctured monogon. If a branch  $b$  of  $\tau$  is adjacent to only one (respectively two distinct) complementary region(s), then there exists an embedded curve  $g$  that hits  $\tau$  efficiently such that  $g \cap \tau$  is contained in  $b$  and consists of one (respectively two) point(s).

On the five-times punctured sphere not every train track is transversely recurrent. An example of a maximal train track on  $\Sigma_{0,5}$  that is recurrent but not transversely recurrent is given in Figure 3.7.

We are interested in birecurrent train tracks because they have good carrying properties – more about this in the next subsection. We finish with two lemmas that simplify the verification of transverse recurrence.

**Lemma 3.3.11.** *A semi-generic recurrent train track  $\tau$  is transversely recurrent if and only if for every large branch  $b$  there exists a curve that hits  $\tau$  efficiently and intersects  $b$ .*

*Proof.* The ‘only if’ statement is immediate. We proceed with the ‘if’ statement. Suppose that  $\tau$  is a semi-generic recurrent train track on  $\Sigma$ . By recurrence, no closed train path on  $\tau$  traverses only mixed branches. Let  $b$  be a branch of  $\tau$  with a small half-branch  $h$ . There exists a large branch  $b'$  and an injective train path

$$p: (0, n) \rightarrow \Sigma$$

on  $\tau$  such that  $p((0, n)) = b$ ,  $p((\frac{1}{2}, 1)) = h$ ,  $p((n-1, n)) = b'$  and  $p((i-1, i))$  is a mixed branch for all  $2 \leq i \leq n-1$ . In fact, such  $b'$  and  $p$  are unique. By assumption there exists a  $C^1$ -embedded curve  $g'$  that hits  $\tau$  efficiently and intersects  $b'$ .

We will isotope  $g'$  along  $p$  to obtain a curve that intersects  $b$ . Write  $b_i$  for the branch  $p((i-1, i))$ , so that  $b_1 = b$  and  $b_n = b'$ . Choose  $t \in (0, n)$  minimal so that  $p(t) \in g'$ . If  $t < 1$  then  $g'$  intersects  $b$  and we are done. Otherwise, take a regular neighbourhood  $N$  of  $p((\frac{1}{2}, t))$  so that  $g'$  intersects  $N$  only in an arc through  $p(t)$ . Replace the intersection arc of  $g$  and  $N$  with the component of  $\text{fr } N - g'$  that intersects  $b$ . This new curve is no longer  $C^1$ , however it is arbitrarily close to a  $C^1$ -embedded curve  $g$ . Observe that  $g$  intersects  $\tau$  efficiently, because  $g'$  does and no new bigons arise in the process. Furthermore  $g$  intersects  $b$ .

Since  $b$  was an arbitrary small or mixed branch of  $\tau$ , we may conclude that  $\tau$  is transversely recurrent.  $\square$

## Measured train tracks

In order to relate train tracks to measured laminations we introduce measures on train tracks.

**Definition 3.3.12** (Measured train track). A *transverse measure*  $m$  on a train track  $\tau$  assigns to every branch  $b$  of  $\tau$  a *weight*  $m(b) \in \mathbb{R}_{\geq 0}$  so that at every switch  $s$  the *switch condition*

$$m(b_1) \dots m(b_k) = m(b_{k+1}) + \dots m(b_n)$$

is satisfied. Here the branches are labeled so that  $\{\{h_1, \dots, h_k\}, \{h_{k+1}, \dots, h_n\}\}$  is the partition of the half-branches incident to  $s$  and  $b_i$  is the branch containing  $h_i$ . The pair  $(\tau, m)$  is called a *measured train track*.

Let  $V(\tau)$  denote the set of all transverse measures on  $\tau$ . In other words,  $V(\tau)$  consists of all  $n$ -vectors of weights that satisfy the switch conditions, where  $n$  denotes the number of branches of  $\tau$ . Every switch condition imposes a linear relation, so  $V(\tau)$  can be identified with the intersection of a linear subspace of  $\mathbb{R}^n$  and the positive orthant. The *dimension* of  $\tau$ , denoted  $\dim(\tau)$ , is the dimension of  $V(\tau)$ . Observe that if  $\tau$  is a semi-generic recurrent train track with  $k$  small branches and  $\ell$  large branches, then  $k - \ell \leq \dim(\tau) \leq k$  because a measure on  $\tau$  is completely determined by the weights it gives to small branches and every large branch imposes a linear relation.

A *projective measure* on a train track  $\tau$  is an equivalence class of transverse measures on  $\tau$  under the equivalence relation  $m \sim rm$  for all  $r \in \mathbb{R}_{>0}$ . Let  $P(\tau)$

denote the collection of all projective measures on  $\tau$ . Note that  $V(\tau)$  has a natural identification with a cone on  $P(\tau)$ . If  $\sigma \subset \tau$  is a birecurrent subtrack, then every measure on  $\sigma$  defines a measure on  $\tau$ . It follows that  $P(\sigma)$  is a non-empty subset of  $P(\tau)$ , called a *face*. For a birecurrent train track  $\tau$ , the family of faces

$$\{P(\sigma) : \sigma \text{ birecurrent subtrack of } \tau\}$$

gives  $P(\tau)$  the structure of a convex polytope of dimension  $\dim(\tau) - 1$  [PeH, Lemma 2.1.2].

**Definition 3.3.13** (Train track polytope).  $P(\tau)$  is the *train track polytope* of  $\tau$ .

Let  $\tau$  be a birecurrent train track and let  $m$  be a measure on  $\tau$ . If  $m$  has integral weights we define a measured lamination  $\Lambda(\tau, m)$  as follows. For every branch  $b$  of  $\tau$  embed  $m(b)$  disjoint parallel smooth arcs in a tie neighbourhood of  $\tau$ , each of which is transverse to the ties, has endpoints on singular ties and is mapped to  $b$  by the tie-collapsing map. Since  $m$  satisfies the switch conditions, we may assume that the endpoints of these arcs match up to a disjoint union of simple closed curves. The corresponding weighted multicurve (with counting measure) is the associated element of  $\mathcal{ML}(\Sigma)$ .

There is a natural way to extend this construction to all measures on  $\tau$  [PeH, Construction 1.7.7], defining a continuous injection  $\Lambda: V(\tau) \rightarrow \mathcal{ML}(\Sigma)$  with image  $\{(\lambda, m) : \lambda \prec \tau\}$  [PeH, Theorem 1.7.12]. It follows from the construction that  $\Lambda(\tau, m_1)$  and  $\Lambda(\tau, m_2)$  represent the same projective lamination if and only if  $m_1 = rm_2$  for some  $r \in \mathbb{R}_{>0}$ . Consequently,  $\Lambda$  induces a continuous injection between quotient spaces

$$P(\tau) \hookrightarrow \mathcal{PML}(\Sigma).$$

This map is an embedding, since  $P(\tau)$  is compact and  $\mathcal{PML}(\Sigma)$  is Hausdorff. From now on we consider  $P(\tau)$  as a subset of  $\mathcal{PML}(\Sigma)$  via this embedding. If  $\tau$  is a complete train track on  $\Sigma$  then  $\text{int } P(\tau)$  is an open subset of  $\mathcal{PML}(\Sigma)$  [PeH, Lemma 3.1.2]. The vertices of the polytope  $P(\tau)$  are called *vertex cycles*. Vertex cycles are curves in  $\mathcal{PML}(\Sigma)$  [MasMi1, §4.1].

A *dumbbell track* is a generic train track with two switches, a large branch connecting the switches and two small branches each incident to only one switch. A *degenerate dumbbell track* is the non-generic track obtained by contracting the large branch of a dumbbell track to a point. The vertex cycles of  $\tau$  correspond to the subtracks of  $\tau$  that are simple closed curves, dumbbell tracks or degenerate dumbbell tracks [Mos, Proposition 3.11.3].

The following theorem is a rephrasing of results in Penner-Harer [PeH, Proposition 2.2.1 and Theorem 2.3.1].

**Theorem 3.3.14.** *Suppose that  $\lambda \in \mathcal{UM}\mathcal{L}(\Sigma)$  is carried by train tracks  $\tau_1$  and  $\tau_2$ . There exists a train track  $\sigma$  such that  $\lambda \prec \sigma$  and  $\sigma \prec \tau_i$  for  $i = 1, 2$ . Moreover, if  $\tau_1$  or  $\tau_2$  is birecurrent then  $\sigma$  is birecurrent as well.*

### Penner-Harer standard train tracks

The space of projective laminations is covered by the train track polytopes of finitely many train tracks. Even more, Penner and Harer construct explicit train tracks whose polytopes equip  $\mathcal{PM}\mathcal{L}(\Sigma)$  with the structure of a CW complex. For simplicity, we will recall their construction only for punctured spheres.

First, we extend the definition of pretracks to surfaces with boundary.

**Definition 3.3.15** (Pretrack with stops). A *pretrack with stops* on  $\Sigma = \Sigma_{g,n}^b$  is an embedded finite graph  $\tau \subset \Sigma$  that satisfies the following conditions. The edges of  $\tau$  are  $C^1$ -arcs with well-defined tangent vectors at the endpoints. For any vertex  $s$  of  $\tau$  there is a unique tangent line  $L \in \text{PT}_s \Sigma$  such that any edge incident to  $v$  is tangent to  $L$  at  $v$ . Every boundary component of  $\Sigma$  contains at most one vertex, called a *boundary switch*. The other vertices are called *regular switches*. The tangent line through a boundary switch is transverse to the boundary. Through every regular switch there is a  $C^1$ -path which is embedded in  $\tau$  and which contains the switch in its interior. All boundary switches have valence 1. Each component of  $\tau$  that is a curve contains a unique bivalent switch. The remaining regular switches have valence at least 3.

For the purpose of this subsection, assume that  $\Sigma = \Sigma_{0,n}$  and that  $\mathfrak{p}$  is a (representative of a) pants decomposition of  $\Sigma$  so that every component of  $\Sigma - \mathfrak{p}$  is bounded by at most two curves of  $\mathfrak{p}$ . For every simple closed curve  $a \in \mathfrak{p}$  take an annular neighbourhood  $N(a)$  of  $a$ , narrow enough so that any two are disjoint. Every closed component of

$$\text{Cut}(\Sigma, \mathfrak{p}) = \Sigma - \bigcup_{a \in \mathfrak{p}} \text{fr } N(a)$$

is a twice punctured disc, a once punctured annulus or an annulus. For each closed component of  $\text{Cut}(\Sigma, \mathfrak{p})$  fix a homeomorphism to some  $\Sigma_{g,n}^b$  where  $\{g, n, b\} \in \{\{0, 2, 1\}, \{0, 1, 2\}, \{0, 0, 2\}\}$ . The pants decomposition  $\mathfrak{p}$  together with a choice of homeomorphism for every component of  $\text{Cut}(\Sigma, \mathfrak{p})$  is called a *basis* of  $\Sigma$ . We

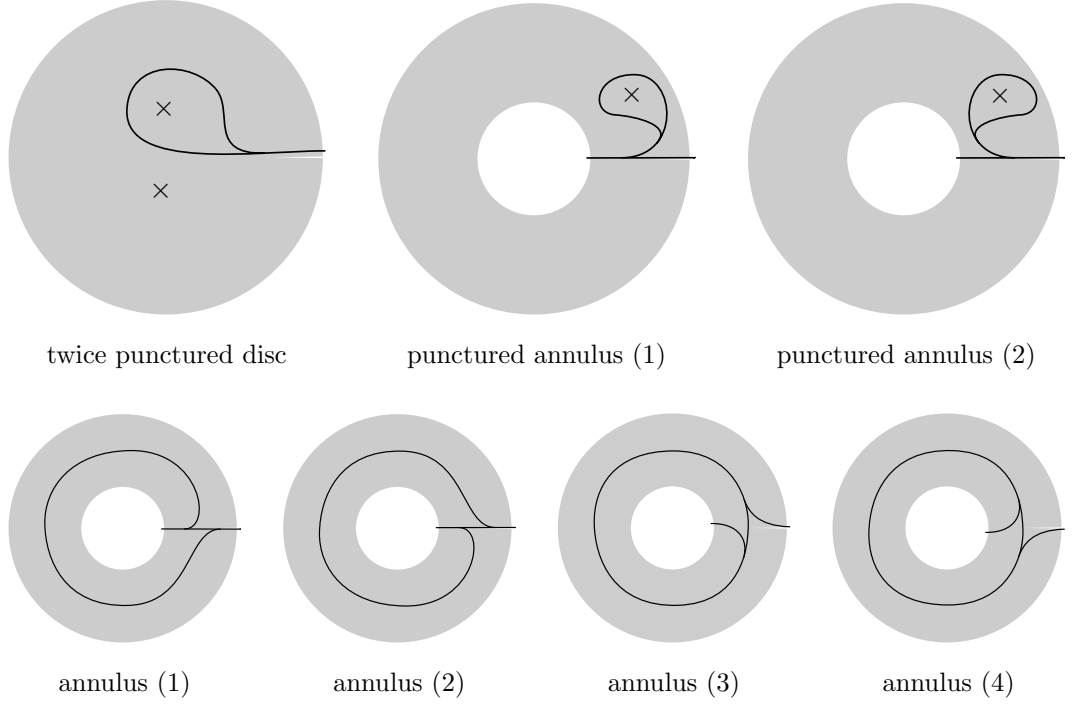


Figure 3.8: Configurations of maximal standard tracks in components of  $\text{Cut}(\Sigma, \mathbf{p})$ .

sometimes leave the collection of homeomorphisms implicit in the notation and write  $\mathbf{p}$  to denote a basis on the pants decomposition  $\mathbf{p}$ .

**Definition 3.3.16** (Standard track). A train track on  $\Sigma$  is called a *Penner-Harer standard train track* (*standard track* for short) with basis  $\mathbf{p}$  if it intersects every component of  $\text{Cut}(\Sigma, \mathbf{p})$  in a subtrack of one of the pretracks with stops depicted in Figure 3.8.

Note that a standard track is maximal if and only if it intersects every component in one of the pretracks with stops of Figure 3.8. There are  $2^{3\xi(\Sigma)-1}$  maximal standard tracks with basis  $\mathbf{p}$ . Any standard track is birecurrent and any maximal standard track is complete [PeH, Lemma 2.6.1].

Using the standard tracks we define a CW complex  $\mathcal{K} = \mathcal{K}(\Sigma)$  as follows. Fix a basis  $x$  of  $\Sigma$  and set  $k = 2^{3\xi(\Sigma)-1}$ . Let  $\tau_1, \tau_2, \dots, \tau_k$  be the maximal standard tracks with basis  $\mathbf{p}$ . We describe an equivalence relation on the set of measured maximal standard tracks. Write  $(\tau_i, m_i) \sim (\tau_j, m_j)$  if the subtrack of  $\tau_i$  where  $m_i > 0$  is the same as the subtrack of  $\tau_j$  where  $m_j > 0$  and on this train track  $m_i$  and  $m_j$  agree. This also induces an equivalence relation on the projective maximal standard tracks.

Define

$$CK = \left( \bigsqcup_{i=1}^k V(\tau_i) \right) / \sim, \quad \mathcal{K} = \left( \bigsqcup_{i=1}^k P(\tau_i) \right) / \sim,$$

using the suggestive notation  $CK$  because this space can be identified with a cone on  $\mathcal{K}$ . Observe that  $\mathcal{K}$  has the structure of a regular CW complex. The  $n$ -cells of  $\mathcal{K}$  are the polytopes of standard tracks of dimension  $n + 1$ . The frontier of the each polytope is a union of faces, which are polytopes of standard tracks of lower dimension. This characterises the attaching maps.

Penner and Harer proved the following result [PeH, Corollary 2.8.6].

**Theorem 3.3.17.** *The inclusions of  $V(\tau_i)$  into  $\mathcal{ML}(\Sigma)$  assemble to a homeomorphism  $CK(\Sigma) \rightarrow \mathcal{ML}(\Sigma)$ .*

**Corollary 3.3.18.** *The inclusions of  $P(\tau_i)$  into  $\mathcal{PML}(\Sigma)$  define a homeomorphism*

$$\mathcal{K}(\Sigma) \rightarrow \mathcal{PML}(\Sigma).$$

*Remark 3.3.19.* It follows that there is an embedding  $\mathcal{PML}(\Sigma) \hookrightarrow \mathcal{ML}(\Sigma)$  that is a section of the projectivisation map  $\Psi$  (defined on page 29). Explicitly, embed  $\mathcal{K}$  into  $CK$  starting with the 0-cells and subsequently extending over the  $n$ -skeleton.

*Example 3.3.20.* We give a description the complex  $\mathcal{K}(\Sigma_{0,4})$ , which only has 4 top-dimensional cells. Let  $a$  be a simple closed curve on  $\Sigma_{0,4}$  and set  $\mathfrak{p} = \{a\}$ . The closed components of  $\text{Cut}(\Sigma, \mathfrak{p})$  are two twice punctured discs and an annulus with core curve  $a$ . The maximal standard tracks  $\tau_1, \tau_2, \tau_3, \tau_4$  depicted in Figure 3.9 all restrict to the same pretrack with stops on the components of  $\text{Cut}(\Sigma, \mathfrak{p})$  that are twice punctured discs. Note that  $\tau_i$  restricts on  $N(a)$  to the configuration ‘annulus (i)’ of Figure 3.8. Now  $\mathcal{K}(\Sigma_{0,4})$  is the frontier of a square with edges  $P(\tau_i)$  and vertices  $P(\tau_i) \cap P(\tau_{i+1})$  where  $i \in \mathbb{Z}/4$ . Clearly  $\mathcal{K}(\Sigma_{0,4}) \cong \mathbb{S}^1$ .

We say that an ending lamination  $\lambda$  is *uniquely ergodic* if  $\Phi^{-1}(\lambda) \subset \mathcal{PML}(\Sigma)$  is a single point. The stable lamination of any pseudo-Anosov mapping class is uniquely ergodic [FatLP, Theorem 12.1].

*Remark 3.3.21.* On  $\Sigma_{0,4}$  every ending lamination is uniquely ergodic. We can deduce this from Example 3.3.20 in the following way. Suppose that there are two distinct measures on  $\lambda \in \mathcal{EL}(\Sigma_{0,4})$ , then there is a non-constant path in  $\Phi^{-1}(\lambda)$ . But the example shows that the complement of the simple closed curves in  $\mathcal{PML}(\Sigma_{0,4})$  is homeomorphic to  $\mathbb{R} - \mathbb{Q}$ . In particular, it is totally disconnected.

On  $\Sigma_{0,5}$  non-uniquely ergodic ending laminations do exist. Explicit examples of such laminations were constructed by Leininger, Lenzhen and Rafi [LeiLR].

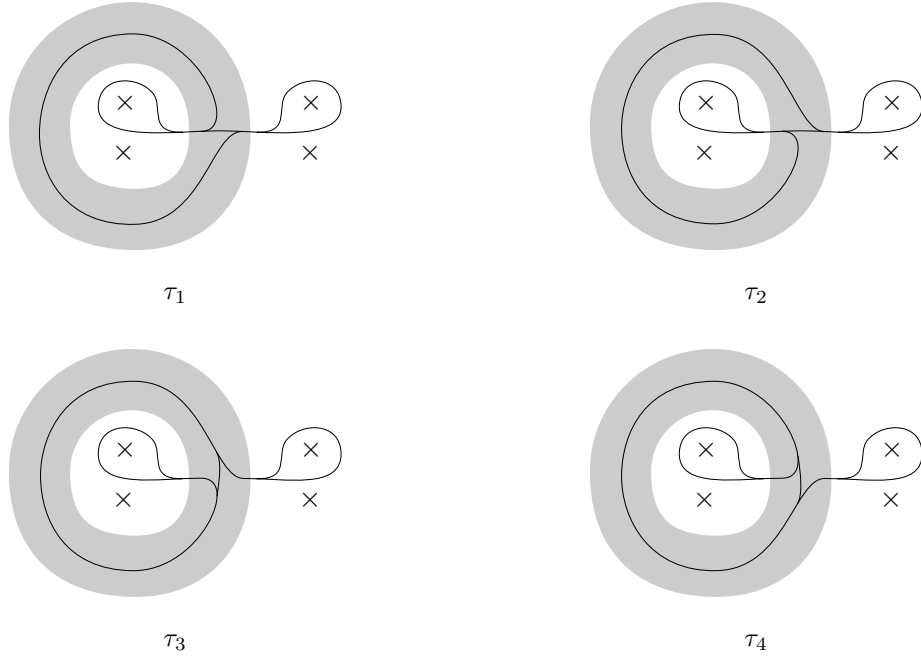


Figure 3.9: Maximal standard tracks on  $\Sigma_{0,4}$ .

### 3.4 Subsurface projection

In this section we recall subsurface projection of curves [MasMi2, §2.3] (see also [Iva1, Iva2]). Subsequently, we generalise subsurface projection to laminations and train tracks.

#### Projecting curves

Let  $W$  be a domain in a surface  $\Sigma$ . We will describe a map that associates to a curve on  $\Sigma$  a collection of curves on  $W$  of bounded diameter. We first pass from a curve on  $\Sigma$  to a collection of curves and arcs on  $W$  (the map  $\rho_W$  below) and then replace the arcs by curves to obtain to a collection of curves on  $W$  (the map  $\pi'_W$  below).

- To  $\alpha \in \mathcal{S}(\Sigma)$  we associate a finite set  $\rho_W(\alpha) \subset \mathcal{S}(W) \cup \mathcal{A}(W)$  as follows. Take a representative  $a$  of  $\alpha$  that intersects every component of  $\partial W$  minimally. If  $a$  is contained in  $W$  we define  $\rho_W(\alpha) = \{[a]\} \subset \mathcal{S}(W)$ . If  $a$  is disjoint from  $W$  then  $\rho_W(\alpha) = \emptyset$ . In the remaining case every component  $\mathbf{a}$  of  $a \cap W$  is an essential arc in  $W$  with endpoints on  $\partial W$ , that is, a representative of an element in  $\mathcal{A}(W)$ . Define

$$\rho_W(\alpha) = \{[\mathbf{a}] \in \mathcal{A}(W) : \mathbf{a} \text{ is a component of } a \cap W\}.$$

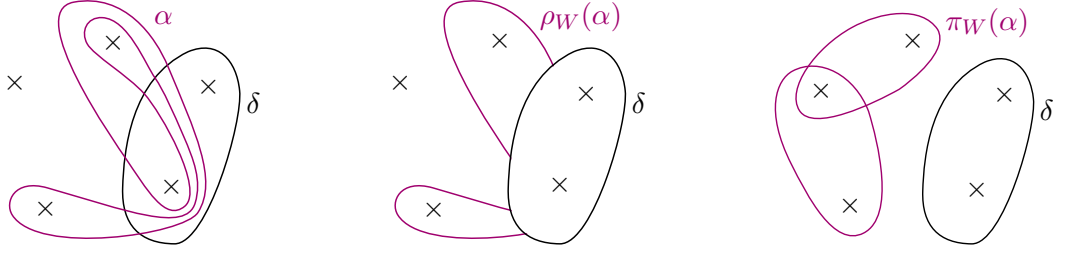


Figure 3.10: Let  $W \subset \Sigma_{0,5}$  be the domain with boundary curve  $\delta$ . The curve  $\alpha \in \mathcal{C}(\Sigma_{0,5})$  (left figure) projects to  $\rho_W(\alpha) \in \mathcal{AC}(W)$  (middle) and  $\pi_W(\alpha) \in \mathcal{C}(W)$  (right).

The definition of  $\rho_W(\alpha)$  is independent of the choice of representative of  $\alpha$ .

- To  $\alpha \in \mathcal{A}(W)$  we associate a finite set  $\pi'_W(\alpha) \subset \mathcal{S}(W)$  as follows. Suppose that  $\mathbf{a} \in \alpha$  is a representative arc and let  $N(\mathbf{a})$  be a regular neighbourhood of  $\mathbf{a}$ . Let  $\pi'_W(\alpha) \subset \mathbb{P}(\mathcal{S}(W))$  be the set of boundary components of  $W - N(\mathbf{a})$  that are essential non-peripheral curves on  $W$ . Note that  $1 \leq |\pi'_W(\alpha)| \leq 2$  and  $\pi'_W$  is independent of the choice of representative of  $\alpha$ .
- For  $\alpha \in \mathcal{S}(W)$  define  $\pi'_W(\alpha) = \{\alpha\}$ .

Write  $\mathbb{P}(S)$  for the power set of a set  $S$ .

**Definition 3.4.1** (Subsurface projection of curves). The assignment

$$\pi_W: \mathcal{S}(\Sigma) \rightarrow \mathbb{P}(\mathcal{S}(W)), \quad \pi_W(\alpha) = \bigcup_{\beta \in \rho_W(\alpha)} \pi'_W(\beta)$$

is called the *subsurface projection* to  $W$ .

Figure 3.10 gives an explicit example of the subsurface projection of a curve. We highlight the following lemma, due to Ivanov [Iva1, Iva2]. Our preferred phrasing appears in [MasMi2, Lemma 2.2].

**Lemma 3.4.2** (Ivanov). *The assignment  $\pi'_W: \mathcal{S}(W) \cup \mathcal{A}(W) \rightarrow \mathbb{P}(\mathcal{S}(W))$  satisfies:*

- (i) *If  $\beta \in \mathcal{S}(W) \cup \mathcal{A}(W)$  then any two curves in  $\{\beta\} \cup \pi'_W(\beta)$  are disjoint.*
- (ii) *If  $\alpha, \beta \in \mathcal{S}(W) \cup \mathcal{A}(W)$  such that  $\iota(\alpha, \beta) = 0$ , then*

$$\text{diam}_{\mathcal{C}(W)}(\pi'_W(\alpha) \cup \pi'_W(\beta)) \leq 2.$$

Ivanov's lemma implies that  $\pi_W$  is 'coarsely Lipschitz' in the sense of the following lemma. Write  $d_\Sigma$  for the distance in  $\mathcal{C}(\Sigma)$ .



**Lemma 3.4.3.** *Any  $\alpha, \beta \in \mathcal{S}(\Sigma)$  satisfy  $\text{diam}_{\mathcal{C}(W)}(\alpha, \beta) \leq 2d_\Sigma(\alpha, \beta) + 2$ .*

*Proof.* If  $d_\Sigma(\alpha, \beta) \leq 1$  then  $\rho_W(\alpha) \cup \rho_W(\beta)$  consists of disjoint arcs, and Lemma 3.4.2 gives that  $d_W(\alpha, \beta) \leq 2$ . The result now follows by induction on  $d_\Sigma(\alpha, \beta)$ .  $\square$

In particular, the subsurface projection to  $W$  of any  $\alpha \in \mathcal{S}(\Sigma)$  is a set of diameter at most 2 in  $\mathcal{C}(W)$ .

*Remark 3.4.4.* To deal with iterated subsurface projections it can be more convenient to define subsurface projection up to bounded distance in the curve graph. Since we will only be concerned with the five-times punctured sphere, whose proper domains have complexity 1, we will not worry about this subtlety.

Note that the mapping class group acts on the collection of domains in  $\Sigma$ , and that for every  $f \in \text{MCG}(\Sigma)$

$$\pi_{fW} = f \circ \pi_W \circ f^{-1}.$$

## Projecting laminations

We generalise subsurface projection of curves to measurable laminations on  $\Sigma$  in the following way. Fix a hyperbolic structure on  $\Sigma$ .

Let  $W, V \subset \Sigma$  be domains. To a lamination  $\lambda \in \mathcal{EL}(V)$  we associate the following set  $\pi_W(\lambda) \subset \mathcal{S}(W) \cup \{\lambda\}$ .

1. If  $\pi_V(\partial W) \neq \emptyset$ , then  $\lambda$  intersects  $W$  in a union of uncountably many arcs of finitely many isotopy types  $\{\alpha_1, \dots, \alpha_n\} \subset \mathcal{A}(W)$ . Define

$$\pi_W(\lambda) = \bigcup_{i=1}^n \pi'_W(\alpha_i).$$

2. If  $\pi_V(\partial W) = \emptyset$  and  $\pi_W(\partial V) \neq \emptyset$  we define  $\pi_W(\lambda) = \pi_W(\partial V)$ .
3. If  $V$  and  $W$  are isotopic we set  $\pi_W(\lambda) = \{\lambda\}$ .
4. Otherwise,  $V$  and  $W$  can be isotoped to be disjoint and we define  $\pi_W(\lambda) = \emptyset$ .

Observe that these four cases are mutually disjoint.

**Definition 3.4.5** (Subsurface projection of laminations). Let  $W \subset \Sigma$  be a domain. Suppose that  $\lambda$  is a measurable lamination on  $\Sigma$  with minimal sublaminations  $\mu_1, \dots, \mu_n$  so that  $\lambda = \bigcup_{i=1}^n \mu_i$ . The *subsurface projection of  $\lambda$  to  $W$*  is the union

$$\pi_W(\lambda) = \bigcup_{i=1}^n \pi_W(\mu_i).$$

It follows from Lemma 3.4.2 that  $\pi_W(\lambda)$  intersects  $\mathcal{C}(W)$  in a set of diameter at most 2. If  $\lambda$  and  $\mu$  are measurable laminations on  $\Sigma$ , we define

$$d_W(\lambda, \mu) = \text{diam}_{\mathcal{C}(W)}(\pi_W(\lambda) \cup \pi_W(\mu)).$$

This generalises the notation  $\pi_\Sigma$  and  $d_\Sigma$  defined on page 22.

### Projecting train tracks

Finally, we define subsurface projection of train tracks.

**Definition 3.4.6** (Subsurface projection of train tracks). Let  $W \subset \Sigma$  be a domain and let  $\tau$  be a train track on  $\Sigma$ . The *subsurface projection*  $\pi_W(\tau) \in \mathbb{P}(\mathcal{S}(W))$  is the union of the sets

$$\{\pi_W(\gamma) : \gamma \text{ is a vertex cycle of } \tau\}.$$

*Remark 3.4.7.* The diameter of  $\pi_W(\tau)$  is bounded by a constant depending on  $\Sigma$ . In fact, since there are only finitely many combinatorial types of train tracks on  $\Sigma$ , there exists a constant  $K = K(\Sigma) > 0$  such that any two vertex cycles  $\alpha, \beta$  of any train track on  $\Sigma$  intersect at most  $K$  times. It then follows from Lemmas 3.1.10 and 3.4.3 that  $d_W(\alpha, \beta) \leq 2d_\Sigma(\alpha, \beta) + 2 \leq 2K + 4$ .

More about subsurface projections of train tracks can be found, for instance, in [MasMoS].

*Remark 3.4.8.* For  $\alpha \in \mathcal{C}(\Sigma_{0,5})$  we write  $\pi_\alpha$  instead of  $\pi_{W_\alpha}$  to denote the subsurface projection to the domain  $W_\alpha \subset \Sigma_{0,5}$  (defined on page 26). This is done solely to avoid double indices.

## Chapter 4

# Structure of $\mathcal{PML}(\Sigma_{0,5})$

Thurston proved that the space of projective laminations on the five-times punctured sphere is homeomorphic to  $\mathbb{S}^3$  [Thu4]. In this chapter we further explore its piecewise-linear structure, coming from train track polytopes.

Section 4.1 investigates particular discs in  $\mathcal{PML}(\Sigma_{0,5})$ . The key results of this section are Corollary 4.1.6 and Proposition 4.1.10.

In Section 4.2 we introduce ‘butterfly train tracks’, which give rise to an explicit  $\Delta$ -complex structure on  $\mathcal{PML}(\Sigma_{0,5})$ . Using this structure, we prove that the extended mapping class group acts on  $\mathcal{PML}(\Sigma_{0,5})$  by orientation-preserving homeomorphisms (Proposition 4.2.7).

Section 4.3 studies the embedding of the pentagon in  $\mathcal{C}(\Sigma_{0,5})$  into  $\mathcal{PML}(\Sigma_{0,5})$ . We show that this loop is an unknot (Proposition 4.3.3) that forms a Hopf link together with the associated decagon (Proposition 4.3.5).

We continue the analysis of the pentagon and its decagon in Section 4.4. Theorem 4.4.6 is the main result of this section and of the chapter. This theorem gives a characterisation of the embedding in  $\mathcal{PML}(\Sigma_{0,5})$  of the induced subgraph in  $\mathcal{C}(\Sigma_{0,5})$  on the pentagon and its decagon. In particular, we find a loop in  $\mathcal{C}(\Sigma_{0,5})$  whose image in  $\mathcal{PML}(\Sigma_{0,5})$  is a trefoil knot (Corollary 4.4.7).

### 4.1 Discs in $\mathcal{PML}(\Sigma_{0,5})$

Recall that a boundary lamination is a minimal lamination on  $\Sigma_{0,5}$  that has more than one leaf (Definition 3.2.11). The set of projective laminations on  $\Sigma_{0,5}$  whose underlying lamination contains a boundary lamination is denoted  $\mathcal{PABL} \subset \mathcal{PML}(\Sigma_{0,5})$  (Definition 3.2.25).

**Definition 4.1.1.** Define an injection

$$\eta: \mathcal{C}(\Sigma_{0,5}) \rightarrow \mathcal{PML}(\Sigma_{0,5})$$

as follows. For  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  set  $\eta(\alpha) = \alpha$  (compare Example 3.2.19). Extend this linearly to edges of  $\mathcal{C}(\Sigma_{0,5})$ . Explicitly, if  $e$  is an edge connecting  $\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})$  and  $x$  is a point on  $e$  at distance  $0 \leq t \leq 1$  from  $\alpha$ , then  $\eta(x) = (1-t)\alpha + t\beta$ .

*Remark 4.1.2.* The map  $\eta$  is continuous when the curve graph is equipped with the CW topology, that is, the quotient topology coming from the disjoint union of its edges. It is not continuous with respect to the metric topology on  $\mathcal{C}(\Sigma_{0,5})$ . For instance, if  $\alpha, \beta, \gamma \in \mathcal{S}(\Sigma_{0,5})$  and  $\iota(\alpha, \beta) = \iota(\alpha, \gamma) = 0$ , then the sequence  $\frac{n-1}{n}\alpha + \frac{1}{n}T_\gamma^{n^2}(\beta)$  converges to  $\alpha$  in metric, but converges to  $\gamma$  in  $\mathcal{PML}(\Sigma_{0,5})$ .

Recall that we write  $\iota(\lambda, \mu) = 0$  if and only if  $\lambda \cup \mu$  is a lamination (page 28).

**Definition 4.1.3.** For  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  we define

$$D_\alpha = \{\lambda \in \mathcal{PML}(\Sigma_{0,5}) : \iota(\alpha, \lambda) = 0\}.$$

Furthermore, if  $\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})$  with  $\iota(\alpha, \beta) = 0$  then we write  $D_{\alpha, \beta} = D_\alpha \cap D_\beta$ .

Recall that  $W_\alpha$  denotes the domain in  $\Sigma_{0,5}$  that has  $\alpha$  as a boundary curve (page 26). Note that  $D_\alpha$  is a cone on the circle  $\mathcal{PML}(W_\alpha)$  with apex  $\alpha$ . If  $e$  is an edge of the curve graph connecting  $\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})$ , then  $\eta(e) = D_{\alpha, \beta}$ . Therefore, the image of  $\eta$  is  $\bigcup_{\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})} D_{\alpha, \beta}$ . Since any measurable lamination on  $\Sigma_{0,5}$  which does not contain a boundary lamination is a multicurve, we find that

$$\mathcal{PABL} = \mathcal{PML}(\Sigma_{0,5}) - \eta(\mathcal{C}(\Sigma_{0,5})).$$

### Unzipping a train track along a carried curve

Let the Euler index of a surface with cusps be as defined in Definition 3.3.6. The following result appears to be well known, however, we are not aware of any written account on this. For completeness we include a proof.

**Lemma 4.1.4.** *Let  $\tau$  be a birecurrent train track on a surface  $\Sigma$  and let  $\alpha$  be a curve carried by  $\tau$ . There exists a pretrack  $\sigma \prec \tau$  so that*

$$\lambda \prec \sigma \iff \lambda \prec \tau \text{ and } \iota(\lambda, \alpha) = 0,$$

*for all  $\lambda \in \mathcal{UML}(\Sigma)$ . One of the complementary components of  $\sigma$  is an annulus with core curve  $\alpha$ . All other components have negative Euler index.*

*Proof.* Let  $N \subset \Sigma$  be a tie neighbourhood of  $\tau$ . Since  $\alpha$  is carried by  $\tau$ , there exists a smooth embedding  $a$  of  $\alpha$  into the interior of  $N$  that is transverse to the ties. Let  $A$  be an open regular neighbourhood of  $a$  that is contained in  $N$ . Every tie of  $N$  intersects  $M = N - A$  in finitely many connected components, which will be the ties of  $M$ . These ties give  $M$  the structure of a tie neighbourhood of some pretrack  $\sigma$ . Since  $\Sigma - M = (\Sigma - N) \cup A$ , every complementary component of  $\sigma$  different from  $A$  has the same Euler index as a complementary component of  $\tau$ .

Note that  $\sigma \prec \tau$ , because  $\sigma \subset N$  and  $\sigma$  is transverse to the ties of  $N$ . Any lamination carried by  $\sigma$  is disjoint from  $\alpha$ , as  $\sigma$  and  $a$  are disjoint subsets of  $\Sigma$ . This proves that if  $\lambda \prec \sigma$  then  $\lambda \prec \tau$  and  $\iota(\lambda, \alpha) = 0$ .

For the converse, suppose that  $\lambda \in \mathcal{UML}(\Sigma)$  is carried by  $\tau$  and  $\iota(\lambda, \alpha) = 0$ . Then  $\lambda \cup \alpha$  defines a lamination carried by  $\tau$ , so there is a carrying map  $\varphi: \Sigma \rightarrow \Sigma$  with  $\varphi(\lambda \cup \alpha) \subset \text{int } N$ . We may assume that  $\varphi(\alpha) = a$ , and therefore  $\varphi(\lambda) \subset \text{int } M$ . The same carrying map  $\varphi$  shows that  $\lambda \prec \sigma$ .  $\square$

We use this lemma to prove the following proposition.

**Proposition 4.1.5.** *Let  $\tau$  be a birecurrent train track on  $\Sigma_{0,5}$  and let  $\alpha$  be a curve carried by  $\tau$ . There exists a birecurrent train track  $\rho \prec \tau$  such that for any  $\lambda \in \mathcal{UML}(\Sigma)$ ,*

$$\lambda \prec \rho \iff \lambda \prec \tau \text{ and } \iota(\lambda, \alpha) = 0.$$

*Proof.* Let  $\sigma \prec \tau$  be the pretrack disjoint from  $\alpha$  found in Lemma 4.1.4. One complementary component of  $\sigma$  is an annulus with core curve  $\alpha$ . In particular,  $\sigma$  is not a train track. We discuss how to upgrade  $\sigma$  to a birecurrent train track.

Let  $\sigma^-$  be the union of all components of  $\sigma$  that can be isotoped to be disjoint from  $W_\alpha$  and define  $\sigma^+ = \sigma - \sigma^-$ . Every complementary component of  $\sigma^+$  in  $\Sigma_{0,5}$  has negative Euler index, hence  $\sigma^+$  is a train track. Let  $\rho$  be the maximal recurrent subtrack of  $\sigma^+$ . Since  $\tau$  is transversely recurrent and  $\rho \prec \tau$ , also  $\rho$  is transversely recurrent. There are no laminations on a pair of pants, so a lamination  $\lambda \in \mathcal{UML}(\Sigma)$  is carried by  $\sigma$  if and only if it is carried by  $\rho$ . Together with Lemma 4.1.4 this proves the proposition.  $\square$

We highlight the following consequence of Proposition 4.1.5.

**Corollary 4.1.6.** *If  $\tau$  is a birecurrent train track  $\tau$  on  $\Sigma_{0,5}$  and  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  is carried by  $\tau$ , then  $D_\alpha$  intersects  $P(\tau)$  in a convex polytope.*

*Proof.* Let  $\tau$  be a birecurrent train track and let  $\alpha$  be a curve on  $\Sigma_{0,5}$  carried by  $\tau$ . According to Proposition 4.1.5, there exists a birecurrent train track  $\rho \prec \tau$  such that

if  $\lambda \in \mathcal{UM}\mathcal{L}(\Sigma_{0,5})$ , then  $\lambda \prec \rho$  if and only if  $\lambda \prec \tau$  and  $\iota(\lambda, \alpha) = 0$ . In other words,  $D_\alpha \cap P(\tau) = P(\rho)$ . By Proposition 3.3.8 there is a finite sequence of splits from a subtrack of  $\tau$  to  $\rho$ . It follows that  $P(\rho)$  can be obtained from  $P(\tau)$  by imposing finitely many linear relations, hence  $P(\rho)$  is a convex polytope in  $P(\tau)$ .  $\square$

*Remark 4.1.7.* It follows that for every  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  the embedding  $D_\alpha \hookrightarrow \mathcal{PM}\mathcal{L}(\Sigma_{0,5})$  is piecewise linear. This was also observed by Gabai [Gab1, Proposition 2.2].

## Clasping discs

The remaining part of Section 4.1 is dedicated to proving Proposition 4.1.10.

We first set up some terminology. The *unit disc* is the subspace  $\{z \in \mathbb{C} : z \leq 1\}$  of  $\mathbb{C}$ . Any space homeomorphic to the unit disc is called a *disc*. An embedded circle  $S \subset \mathbb{S}^3$  is called an *unknot* if there exists an embedded disc that has  $S$  as its frontier. A union of disjointly embedded circles is called an *unlink* if there exist disjointly embedded discs that have the circles as their frontiers. Let  $S \subset \mathbb{S}^3$  be an unknot. A regular neighbourhood of  $S$  is a solid torus. Let  $T$  be a curve on the frontier of this torus that bounds a disc in its interior. A union of disjointly embedded circles isotopic to  $S \cup T$  is called a *Hopf link*.

Let  $D, D'$  be two copies of the unit disc and consider the quotient

$$C = D \sqcup D' / \sim$$

where  $(z \in D) \sim (w \in D')$  if and only if  $z \in I \subset \mathbb{R} \subset \mathbb{C}$  and  $w = z - 1$ . Intuitively,  $D$  and  $D'$  are glued along a radius in such a way that the frontier circles are disjoint.

**Definition 4.1.8.** Given an embedding of  $C$  into  $\mathbb{R}^3$  (or  $\mathbb{S}^3$ ).

- We say that  $D$  and  $D'$  *clasp* if, up to isotopy,

$$D = \{(x, y, 0) : x^2 + y^2 \leq 1\}, \quad D' = \{(x, 0, z) : (x - 1)^2 + z^2 \leq 1\}.$$

- We say that  $D$  and  $D'$  *touch* if, up to isotopy,

$$\begin{aligned} D &= \{(x, y, 0) : x^2 + y^2 \leq 1, y \geq 0\} \cup \{(x, z, 0) : x^2 + z^2 \leq 1, z \leq 0\}, \\ D' &= \{(x, y, 0) : (x - 1)^2 + y^2 \leq 1, y \leq 0\} \cup \{(x, z, 0) : (x - 1)^2 + z^2 \leq 1, z \geq 0\}. \end{aligned}$$

These two embeddings of  $C$  are illustrated in Figures 4.1 and 4.2 .

*Remark 4.1.9.* Every embedding of  $C$  into  $\mathbb{R}^3$  is isotopic to one of the two embeddings given in Definition 4.1.8. To see this, note that  $C$  has the structure of a 2-dimensional

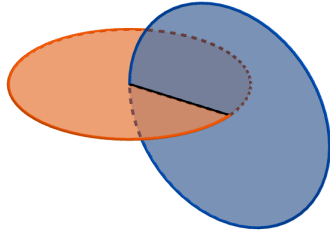


Figure 4.1: Clasping discs.

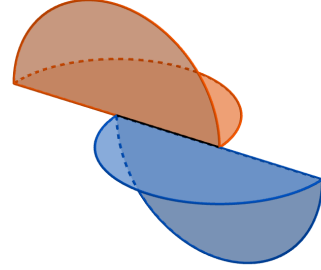


Figure 4.2: Touching discs.

CW complex with 2 vertices, 3 edges and 2 faces. Since  $C$  is contractible, any ‘thickening’ of  $C$  to a 3-manifold is a ball. It follows that the embeddings of  $C$  into  $\mathbb{R}^3$  correspond to the thickenings of  $C$ . The possible thickenings of  $C$  are determined by the embeddings of the links of the vertices of  $C$  into  $\mathbb{S}^2$  which induce compatible thickenings over its edges and faces. The link of a vertex of  $C$  is a connected graph on 3 vertices of valence 1, 1 and 4. This graph only has two isotopy classes of embeddings into  $\mathbb{S}^2$ . Both vertices of  $C$  need to have the same embedded link for the embedding to extend over the edges of  $C$ . Consequently,  $C$  has exactly two distinct thickenings.

Observe that  $D$  and  $D'$  touch if and only if  $\text{fr}(D) \cup \text{fr}(D')$  is an unlink;  $D$  and  $D'$  clasp if and only if  $\text{fr}(D) \cup \text{fr}(D')$  is a Hopf link.

We will use this terminology to describe discs in  $\mathcal{PML}(\Sigma_{0,5}) \cong \mathbb{S}^3$ . If  $\alpha$  and  $\beta$  are disjoint curves on the five-times punctured sphere, then  $D_\alpha$  and  $D_\beta$  are discs that intersect along a radius. We can ask whether these discs clasp or touch.

**Proposition 4.1.10.** *If  $\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})$  are disjoint, then the discs  $D_\alpha$  and  $D_\beta$  clasp.*

*Proof.* The action of  $\text{MCG}(\Sigma_{0,5})$  on  $\mathcal{C}(\Sigma_{0,5})$  is transitive on edges. Furthermore,  $\text{MCG}(\Sigma_{0,5})$  acts on  $\mathcal{PML}(\Sigma_{0,5})$  by homeomorphisms, so clasping discs map to clasping discs. Therefore, it suffices to find one pair of curves  $\{\alpha, \beta\}$  for which  $D_\alpha$  and  $D_\beta$  clasp.

Let  $\tau$  be a complete train track on  $\Sigma_{0,5}$  and let  $P(\tau)$  be its train track polytope (Definition 3.3.13). Recall that the interior of  $P(\tau)$  in  $\mathcal{PML}(\Sigma_{0,5})$  is a non-empty open set. Since curves are dense in  $\mathcal{PML}(\Sigma_{0,5})$ , there exists a curve  $\alpha$  that is fully carried by  $\tau$ . Then  $D_\alpha \cap P(\tau)$  is a neighbourhood of  $\alpha$  in  $D_\alpha$ , so every lamination  $\lambda$  with  $\iota(\lambda, \alpha) = 0$  is carried by  $\tau$ . In particular,  $D_\alpha \subset P(\tau)$ . Let  $\beta$  be a curve disjoint from  $\alpha$ . Then  $\beta \prec \tau$  and by Corollary 4.1.6 both  $D_\alpha$  and  $D_\beta$  intersect  $P(\tau)$  in a convex polytope. Since  $\alpha \in D_{\alpha,\beta}$  is an interior point of  $P(\tau)$ , these polytopes must

be 2-dimensional. Hence  $D_\alpha$  and  $D_\beta \cap P(\tau)$  are polygons in intersecting planes in  $P(\tau)$ , and the interiors of these polygons have non-empty intersection. We conclude that  $D_\alpha$  and  $D_\beta$  are clasping discs.  $\square$

## 4.2 Butterfly train tracks

This section introduces a particular collection of train tracks, called ‘butterflies’. We use these train tracks to find a  $\Delta$ -complex structure on  $\mathcal{PML}(\Sigma_{0,5})$  and to prove Proposition 4.2.7.

**Definition 4.2.1** (Butterflies). The train tracks  $\tau_1, \tau_2, \tau_3, \tau_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4$  on  $\Sigma_{0,5}$  depicted in Figure 4.3 are called the *butterfly train tracks*.

Note that, for every  $1 \leq i \leq 4$ , the train tracks  $\tau_i$  and  $\sigma_i$  are maximal and recurrent. Using Lemma 3.3.11 it is not hard to observe that they are also transversely recurrent.

*Remark 4.2.2.* To simplify our figures, we depict curves on  $\Sigma_{0,5}$  in the following way. Instead of drawing a curve  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  itself, we draw the unique arc connecting punctures of  $\Sigma_{0,5}$  such that  $\alpha$  bounds a small neighbourhood of that arc.

Let  $\gamma_1, \gamma_2, \gamma_4, \beta$  and  $\alpha$  be the curves on  $\Sigma_{0,5}$  depicted in Figure 4.4. Observe that for every  $1 \leq i \leq 4$ ,  $P(\tau_i)$  is a tetrahedron with vertex cycles  $\{\gamma_1, \gamma_2, \gamma_4, \beta\}$  and  $P(\sigma_i)$  is a tetrahedron with vertex cycles  $\{\gamma_1, \gamma_2, \gamma_4, \alpha\}$ . Furthermore, there is a reflection  $r \in \text{MCG}(\Sigma_{0,5})$  such that  $r(\tau_1) = \sigma_1$ ,  $r(\tau_2) = \sigma_4$ ,  $r(\tau_3) = \sigma_3$  and  $r(\tau_4) = \sigma_2$ .

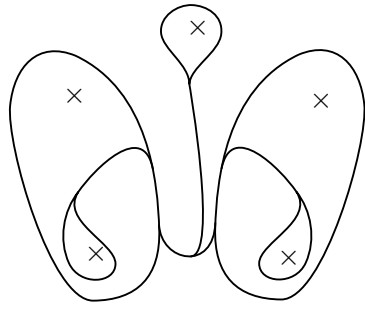
Fix the basis  $\mathfrak{p} = \{\gamma_1, \gamma_2\}$  on  $\Sigma_{0,5}$ . In Corollary 3.3.18 we described a regular CW structure  $\mathcal{K}$  on  $\mathcal{PML}(\Sigma_{0,5})$  whose 3-cells are the polytopes of maximal Penner-Harer standard train tracks (Definition 3.3.16). This structure has 32 3-cells. We use the following lemma to partition these 3-cells into eight 4-tuples, each corresponding to one butterfly train track.

**Lemma 4.2.3.** *Each of the four train tracks that can be obtained from a butterfly by performing a left or right split on both large branches is a standard track with basis  $\mathfrak{p}$ . Conversely, every maximal standard train track with basis  $\mathfrak{p}$  folds to a unique butterfly.*

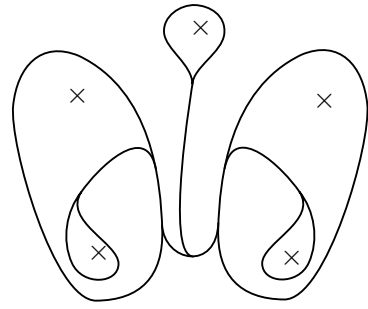
*Proof.* This is immediate from the definition of the standard tracks. Compare Figure 3.8 to Figure 4.3.  $\square$

As a consequence of Lemma 4.2.3, the polytopes of the butterfly train tracks cover all of  $\mathcal{PML}(\Sigma_{0,5})$ . We will prove that these polytopes intersect in faces.

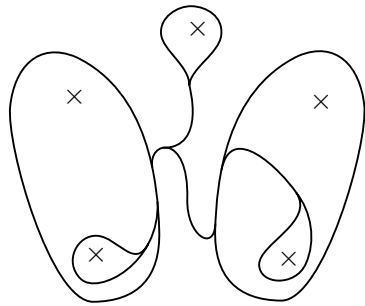




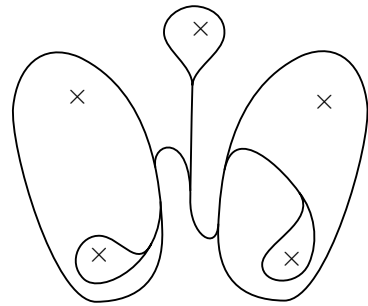
$\tau_1$



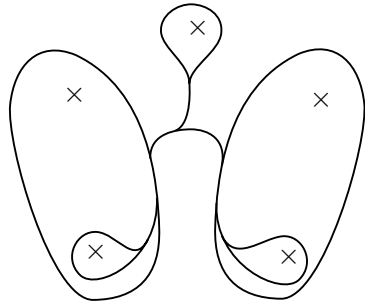
$\sigma_1$



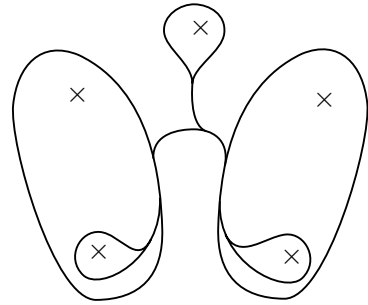
$\tau_2$



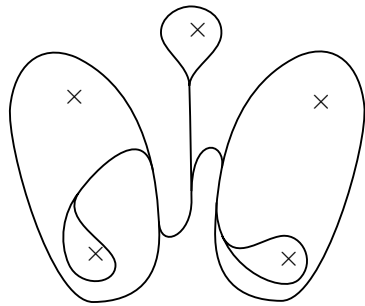
$\sigma_2$



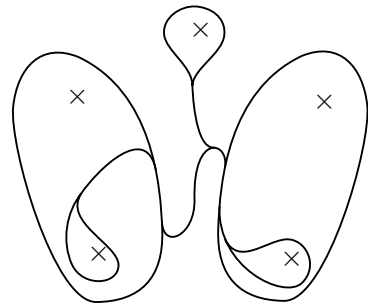
$\tau_3$



$\sigma_3$



$\tau_4$



$\sigma_4$

Figure 4.3: The eight butterfly train tracks.

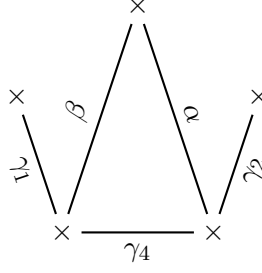


Figure 4.4: For  $1 \leq i \leq 4$ , the vertex cycles of  $\tau_i$  are the curves (represented by arcs)  $\{\gamma_1, \gamma_2, \gamma_4, \beta\}$  and the vertex cycles of  $\sigma_i$  are the curves  $\{\gamma_1, \gamma_2, \gamma_4, \alpha\}$ .

### A $\Delta$ -complex structure on $\mathcal{PML}(\Sigma_{0,5})$

We will use the polytopes of butterfly train tracks to give  $\mathcal{PML}(\Sigma_{0,5})$  the structure of a  $\Delta$ -complex (defined as in [Hat1, p.102 and 533]). Informally, a  $\Delta$ -complex is a generalisation of a simplicial complex, dropping the requirement that simplices are uniquely determined by their vertices. We build this  $\Delta$ -complex from a collection of disjoint tetrahedra by describing the face-identifications.

Let  $\mathfrak{B}$  be the collection of all butterfly train tracks and their birecurrent subtracks. There is an abstract 3-dimensional  $\Delta$ -complex  $\mathcal{W}$  whose  $n$ -simplices correspond to train tracks in  $\mathfrak{B}$  of dimension  $n + 1$  and whose attaching maps correspond to taking subtracks. In particular:

- $\mathcal{W}$  has 5 vertices  $G_1, G_2, G_4, B, A$ , corresponding to  $\gamma_1, \gamma_2, \gamma_4, \beta, \alpha$ , respectively.
- $\mathcal{W}$  has 13 edges: one edge connecting each pair  $\{G_1, A\}$ ,  $\{A, G_4\}$ ,  $\{G_4, B\}$ ,  $\{B, G_2\}$ ,  $\{G_2, G_1\}$  and two edges connecting each pair  $\{G_1, G_4\}$ ,  $\{G_1, B\}$ ,  $\{G_2, A\}$ ,  $\{G_2, G_4\}$ . This gives the 1-skeleton illustrated in Figure 4.5.
- $\mathcal{W}$  has 16 triangles: two for each triple  $\{\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}\}$ ,  $\{\mathbf{G}_1, \mathbf{G}_2, \mathbf{A}\}$ ,  $\{G_1, G_4, B\}$ ,  $\{G_2, G_4, A\}$ ,  $\{G_2, G_4, B\}$ ,  $\{G_1, G_4, B\}$  and four for the triple  $\{\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_4\}$ .
- $\mathcal{W}$  has 8 tetrahedra: one corresponding to each butterfly train track. Write  $T_i$  (respectively  $S_i$ ) for the tetrahedron corresponding to  $\tau_i$  (respectively  $\sigma_i$ ).

To get a better understanding of this complex, we discuss how to obtain  $\mathcal{W}$  by gluing its tetrahedra. Note that all tetrahedra of  $\mathcal{W}$  meet the edge  $\{G_1, G_2\}$ . Performing first the gluings of the eight 2-faces that meet this edge (in **bold**), we obtain a octagonal bipyramid with *top vertex*  $G_2$  and *bottom vertex*  $G_1$ . Figure 4.6 illustrates this bipyramid, viewed from the top (from the outside) and bottom (from the inside).

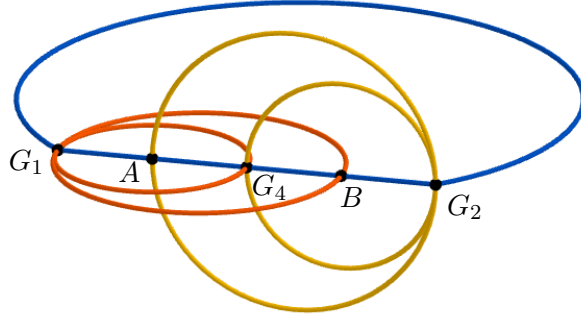


Figure 4.5: The 1-skeleton of the  $\Delta$ -complex  $\mathcal{W}$ . Here the yellow circles lie in a vertical plane and the orange circles lie in a horizontal plane.

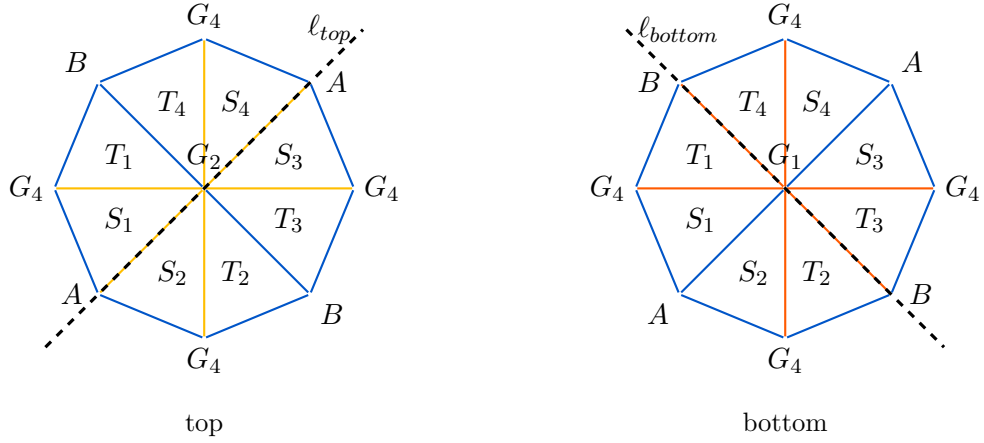


Figure 4.6: The octagonal bipyramid that glues to  $\mathcal{W}$  viewed from the top and bottom.

The remaining gluings pair faces incident to the top vertex, and similarly for faces incident to the bottom vertex. Specifically, faces on the top are identified with their image under reflection in the line  $\ell_{top}$  through  $G_2$  and  $A$ . The faces incident to the top vertex are glued to their image under the reflection in the plane  $\ell_{bottom}$  through  $G_1$  and  $B$ . See Figure 4.6.

We claim that  $\mathcal{W}$  is homeomorphic to  $\mathbb{S}^3$ . This can be seen as follows. The octagonal bipyramid with its face identifications is homeomorphic to a single tetrahedron  $[v_0v_1v_2v_3]$  with face identifications  $[v_0v_1v_2] \sim [v_0v_1v_3]$  and  $[v_0v_2v_3] \sim [v_1v_2v_3]$  (informally, this homeomorphism ‘forgets’ the vertices  $G_1, G_2, G_4$  and merges simplices incident to these vertices). Cutting this tetrahedron along a plane passing through the midpoints of four edges gives the standard decomposition of  $\mathbb{S}^3$  as the union of two solid tori. Compare [Hat1, Exercise 7 of §2.1].

We will prove in Proposition 4.2.5 below that  $\mathcal{W}$  assigns a  $\Delta$ -complex structure

to  $\mathcal{PML}(\Sigma_{0,5})$ . The proof of this proposition relies on the following classical theorem, known as ‘Invariance of Domain’ [Brou]. A more recent account can be found, for instance, in [Hat1, Theorem 2B.3].

**Theorem 4.2.4** (Brouwer). *If  $V$  is an open set in  $\mathbb{R}^n$  and  $h: V \rightarrow \mathbb{R}^n$  is an injective continuous map, then  $h(V)$  is open in  $\mathbb{R}^n$ .*

**Proposition 4.2.5.** *Let  $F$  be the continuous map*

$$F: \mathcal{W} \rightarrow \mathcal{PML}(\Sigma_{0,5})$$

*such that for any  $\tau \in \mathfrak{B}$  the restriction of  $F$  to the simplex corresponding to  $\tau$  is a homeomorphism to  $P(\tau) \subset \mathcal{PML}(\Sigma_{0,5})$ . Then  $F$  is a homeomorphism.*

*Proof.* Write  $\mathcal{W}^n$  for the  $n$ -skeleton of  $\mathcal{W}$ . Recall that a map  $f: X \rightarrow Y$  is a *local homeomorphism* if every  $x \in X$  has an open neighbourhood  $N \subset X$  such that  $f(N)$  is an open subset of  $Y$  and  $f|_N: N \rightarrow f(N)$  is a homeomorphism.

**Claim.** *If  $F|_{\mathcal{W}-\mathcal{W}^n}$  is injective, then  $F|_{\mathcal{W}-\mathcal{W}^{n-1}}$  is a local homeomorphism.*

*Proof of claim.* Suppose that  $x \in \mathcal{W} - \mathcal{W}^{n-1}$ . Let  $N \subset \mathcal{W} - \mathcal{W}^{n-1}$  be an open neighbourhood of  $x$  in  $\mathcal{W}$  that intersects at most one  $n$ -simplex. Any two points in  $N$  which are not contained in one simplex are contained in  $\mathcal{W} - \mathcal{W}^n$ . Since  $F$  is injective when restricted to a simplex or to  $\mathcal{W} - \mathcal{W}^n$ , it follows that  $F|_N$  is injective. Since  $N$  is homeomorphic to an open set in  $\mathbb{R}^3$  and  $\mathcal{PML}(\Sigma_{0,5}) \cong \mathbb{S}^3$ , we can apply Invariance of Domain. This implies that  $F(N)$  is an open subset of  $\mathcal{PML}(\Sigma_{0,5})$  and  $F|_N: N \rightarrow F(N)$  is a homeomorphism.  $\square$

We also prove the following general claim.

**Claim.** *Given a Hausdorff space  $X$  and a local homeomorphism  $f: X \rightarrow Y$ . If there is an open dense set  $V \subset X$  such that  $f|_V$  is injective, then  $f$  is injective.*

*Proof of claim.* Suppose that  $x_1, x_2 \in X$  are distinct points with  $f(x_1) = f(x_2) = y$ . Since  $X$  is Hausdorff and  $f$  is a local homeomorphism, there exist disjoint open neighbourhoods  $N_i$  of  $x_i$  and an open neighbourhood  $N$  of  $y$  such that  $f$  restricts to a homeomorphism from  $N_i$  to  $N$ , for each  $i = 1, 2$ . Define  $V_i = V \cap N_i$ . Then  $V_i$  is an open dense subset of  $N_i$ , so  $f(V_i)$  is an open dense subset of  $N$ . Since  $f|_V$  is injective,  $f(V_1)$  and  $f(V_2)$  are disjoint. It follows that  $\text{cl}(f(V_1))$  is disjoint from  $f(V_2)$ , contradicting that  $f(V_1)$  is dense in  $N$ . This shows that  $f$  is injective.  $\square$

Lemma 4.2.3 implies that the interiors of butterfly polytopes are disjoint. In other words,  $F|_{\mathcal{W}-\mathcal{W}^2}$  is injective. We now apply the two claims alternatingly to show that  $F$  is an injective local homeomorphism. By the first claim the map  $F|_{\mathcal{W}-\mathcal{W}^1}$  is a local homeomorphism, and by the second claim it is injective. By the first claim the map  $F|_{\mathcal{W}-\mathcal{W}^0}$  is a local homeomorphism, and by the second claim it is injective. Finally, by the first claim the map  $F$  is a local homeomorphism, and by the second claim it is injective.

On the other hand, Lemma 4.2.3 implies that  $F$  is surjective. Hence  $F$  is a bijective local homeomorphism, so it is a homeomorphism.  $\square$

In particular, Proposition 4.2.5 implies that the polytopes of butterfly train tracks intersect only in faces corresponding to common subtracks.

### The action of the extended mapping class group on $\mathcal{PML}(\Sigma_{0,5})$

The goal of this section is to prove Proposition 4.2.7. This says that the extended mapping class group acts on  $\mathcal{PML}(\Sigma_{0,5})$  by orientation-preserving homeomorphisms.

Fix a curve  $\alpha \in \mathcal{S}(\Sigma_{0,5})$ . Recall that  $H_\alpha$  denotes the half Dehn twist about  $\alpha$  (Example 3.1.1). The mapping class group of  $\Sigma_{0,5}$  is generated by finitely many half Dehn twists (Corollary 3.1.3). Consequently, the extended mapping class group of  $\Sigma_{0,5}$  is generated by finitely many half Dehn twists and one reflection.

Suppose that  $D$  is a PL embedded disc in  $\mathbb{S}^3$  (endowed with a PL structure). Any interior point  $x$  of  $D$  has a neighbourhood basis in  $\mathbb{S}^3$  of neighbourhoods  $N$  for which  $N - D$  has two connected components. We will call these components the *sides* of  $D$ .

**Lemma 4.2.6.** *Let  $U \subset \mathcal{PML}(\Sigma_{0,5})$  be a neighbourhood of  $\alpha$  such that  $U - D_\alpha$  has two components. There is one component  $U_{top}$  of  $U - D_\alpha$  such that for any  $\lambda \in \mathcal{PML}(\Sigma_{0,5}) - D_\alpha$  the sequence  $(H_\alpha^i(\lambda))_{i \in \mathbb{N}}$  is eventually contained in  $U_{top}$ .*

*Proof.* Let  $U \subset \mathcal{PML}(\Sigma_{0,5})$  be a neighbourhood of  $\alpha$  such that  $U - D_\alpha$  has two components. Suppose that  $\lambda \in \mathcal{PML}(\Sigma_{0,5}) - D_\alpha$  and define  $\lambda_i = H_\alpha^i(\lambda)$ . In Example 3.2.23 we noted that  $(\lambda_i)_{i \in \mathbb{N}}$  converges to  $\alpha$ .

We first show that  $(\lambda_i)_{i \in \mathbb{N}}$  is eventually contained in one component of  $U - D_\alpha$ . Let  $p_0$  be a path in  $\mathcal{PML}(\Sigma_{0,5}) - D_\alpha$  connecting  $\lambda_0$  and  $\lambda_1$ . Define  $p_i = H_\alpha^i \circ p$  and note that there exists  $M \in \mathbb{N}$  such that  $p_i \subset U$  for all  $i \geq M$ . The concatenation  $p_M * p_{M+1} * \cdots * p_{i-1}$  is a path in  $U - D_\alpha$  from  $\lambda_M$  to  $\lambda_i$ , for  $i \geq M$ . In particular,  $(\lambda_i)$  is eventually contained in one connected component of  $U - D_\alpha$ . Denote this component by  $U_{top}$ .

It follows similarly that  $(H_\alpha^i(\mu))_{i \in \mathbb{N}}$  is eventually contained in  $U_{top}$  for any  $\mu \in \mathcal{PML}(\Sigma_{0,5})$ . Choose a path  $p$  connecting  $\lambda$  and  $\mu$  that avoids  $D_\alpha$ . The forward iterates of this path under  $H_\alpha$  are paths disjoint from  $D_\alpha$  whose endpoints approach  $\alpha$ . Eventually these paths are contained in  $U$ .  $\square$

We are now able to give a combinatorial proof of the following fact.

**Proposition 4.2.7.** *The extended mapping class group acts on  $\mathcal{PML}(\Sigma_{0,5})$  by orientation-preserving homeomorphisms.*

*Proof.* The extended mapping class group is generated by half Dehn twists and any one reflection. Lemma 4.2.6 shows that for any  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  the homeomorphism  $H_\alpha$  of  $\mathcal{PML}(\Sigma_{0,5})$  fixes  $D_\alpha$  pointwise and preserves each side of  $D_\alpha$ . Therefore,  $H_\alpha$  is an orientation-preserving homeomorphism of  $\mathcal{PML}(\Sigma_{0,5})$ .

It remains to find one reflection that preserves orientation on  $\mathcal{PML}(\Sigma_{0,5})$ . Consider the butterfly train tracks  $\tau_1$  and  $\sigma_1$  from Figure 4.3. Let  $r$  be the reflection of  $\Sigma_{0,5}$  that takes  $\tau_1$  to  $\sigma_1$ . Specifically,  $r$  fixes the common vertex cycle  $\gamma_4$ ,  $r(\gamma_1) = \gamma_2$  and  $r(\beta) = \alpha$ . It follows that  $r$  preserves orientation on  $\mathcal{PML}(\Sigma_{0,5})$ .  $\square$

*Remark 4.2.8.* Proposition 4.2.7 holds more generally, for every surface  $\Sigma$  of even complexity. We sketch a (more abstract) proof, using Teichmüller space  $\text{Teich}(\Sigma)$ . Fix a reflection  $r: \Sigma \rightarrow \Sigma$  and a pants decomposition  $\mathbf{p}$  such that  $r(\alpha) = \alpha$  for all  $\alpha \in \mathbf{p}$ . Fenchel-Nielsen coordinates determine a homeomorphism

$$\text{Teich}(\Sigma) \cong \mathbb{R}_{>0}^{\xi(\Sigma)} \times \mathbb{R}^{\xi(\Sigma)},$$

where the first  $\xi(\Sigma)$  coordinates are ‘length coordinates’ of the curves in  $\mathbf{p}$  and the last  $\xi(\Sigma)$  coordinates are ‘twist coordinates’ (see, for instance, [FarM, §10.6]).

A (half) Dehn twist  $f \in \{H_\alpha, T_\alpha\}$  about a curve  $\alpha \in \mathbf{p}$  fixes the length coordinates. Furthermore,  $f$  fixes all but one twist coordinate, and on this coordinate it acts as a translation. It follows that  $f$  is an orientation-preserving homeomorphism of  $\text{Teich}(\Sigma)$ . The reflection  $r$  acts on  $\text{Teich}(\Sigma)$  by fixing the length coordinates and inverting every twist coordinate. Therefore,  $r$  preserves the orientation on  $\text{Teich}(\Sigma)$  exactly when  $\xi(\Sigma)$  is even.

Using Thurston’s interpretation of  $\mathcal{PML}(\Sigma)$  as a compactification of  $\text{Teich}(\Sigma)$  [Thu4, Theorem 3], we conclude that  $\text{MCG}(\Sigma)$  acts on  $\mathcal{PML}(\Sigma)$  by orientation-preserving homeomorphisms. The extended mapping class group of  $\Sigma$  acts by orientation-preserving homeomorphisms if and only if  $\xi(\Sigma)$  is even.

### 4.3 Pentagons of curves

The goal of this section is to prove that the pentagon in  $\mathcal{C}(\Sigma_{0,5})$  is an unknot in  $\mathcal{PML}(\Sigma_{0,5})$  (Proposition 4.3.3). Moreover, we show that the pentagon and its decagon form a Hopf link in  $\mathcal{PML}(\Sigma_{0,5})$  (Proposition 4.3.5)

**Definition 4.3.1** (Pentagon). A *pentagon*  $\Gamma$  in  $\mathcal{C}(\Sigma_{0,5})$  is a subgraph with 5 vertices that is homeomorphic to a circle.

Since the curve graph of the five-times punctured sphere has no triangles, a pentagon is the induced subgraph of  $\mathcal{C}(\Sigma_{0,5})$  on its vertex set. Up to the action of the mapping class group there is only one pentagon in  $\mathcal{C}(\Sigma_{0,5})$  [Luo, Lemma 4.2].

Recall that  $\eta$  is the inclusion of  $\mathcal{C}(\Sigma_{0,5})$  into  $\mathcal{PML}(\Sigma_{0,5})$  (Definition 4.1.1).

*Remark 4.3.2.* If  $\Gamma \subset \mathcal{C}(\Sigma_{0,5})$  is a pentagon, we write  $\Gamma_{\mathcal{S}} \subset \mathcal{S}(\Sigma_{0,5})$  for the vertex set of  $\Gamma$ . We call each of  $\Gamma_{\mathcal{S}}, \Gamma$  and  $\eta(\Gamma)$  a *pentagon*.

The collection of curves  $\Gamma_{\mathcal{S}} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$  depicted in Figure 4.9 is an example of a pentagon. Note that any two vertices of a pentagon that are at distance 2 in  $\mathcal{C}(\Sigma_{0,5})$  represent curves that intersect twice.

**Proposition 4.3.3.** *If  $\Gamma \subset \mathcal{C}(\Sigma)$  is a pentagon, then  $\eta(\Gamma) \subset \mathcal{PML}(\Sigma_{0,5})$  bounds a disc.*

*Proof.* Since there is only one pentagon up to the action of  $\text{MCG}(\Sigma_{0,5})$  and  $\text{MCG}(\Sigma_{0,5})$  acts on  $\mathcal{PML}(\Sigma_{0,5})$  by homeomorphisms, it suffices to find one pentagon that bounds a disc. The proof can be outlined as follows. Let  $\tau_1$  and  $\sigma_1$  be the butterfly train tracks illustrated in Figure 4.3. It follows from Proposition 4.2.5 that  $P(\tau_1) \cup P(\sigma_1)$  is a 3-dimensional ball in  $\mathcal{PML}(\Sigma_{0,5})$ . We will find a pentagon in the frontier of  $P(\tau_1) \cup P(\sigma_1)$ . This pentagon is an embedded piecewise linear circle on a 2-sphere, so it bounds a disc.

Let  $b_1, b_2, \dots, b_6$  (respectively  $a_1, a_2, \dots, a_6$ ) denote the small branches of  $\tau_1$  (respectively  $\sigma_1$ ), labelled as in Figure 4.7 (respectively Figure 4.8). Write  $w_k \in \mathbb{R}_{>0}$  (respectively  $v_k$ ) for the weight of a lamination carried by  $\tau_1$  (respectively  $\sigma_1$ ) on the branch  $b_k$  (respectively  $a_k$ ). The two large branches of  $\tau$  give the equations

$$w_6 = 2w_3, \quad 2w_2 = 2w_5 + w_6,$$

so we can eliminate  $w_6$  and  $w_2$  and use  $\{[w_1 : w_3 : w_4 : w_5] \mid w_i \in \mathbb{R}_{\geq 0}\}$  as projective coordinates for  $P(\tau_1)$ . In these coordinates, the vertex cycles of the tetrahedron  $P(\tau)$  are  $\gamma_1 = [1 : 0 : 0 : 0]$ ,  $\gamma_4 = [0 : 1 : 0 : 0]$ ,  $\gamma_2 = [0 : 0 : 1 : 0]$  and  $\beta = [0 : 0 : 0 : 1]$ .

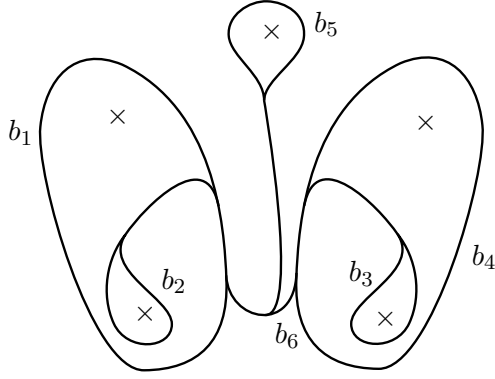


Figure 4.7: The butterfly  $\tau_1$  with small branches  $b_i$ ,  $1 \leq i \leq 6$ .

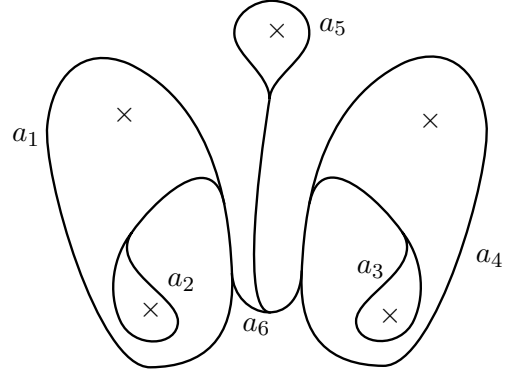


Figure 4.8: The butterfly  $\sigma_1$  with small branches  $a_i$ ,  $1 \leq i \leq 6$ .

Similarly, for  $\sigma_1$  we find the equations

$$v_6 = 2v_2, \quad 2v_3 = 2v_5 + v_6,$$

so we eliminate  $v_6$  and  $v_3$  and use  $\{[v_1 : v_2 : v_4 : v_5] \mid v_i \in \mathbb{R}_{\geq 0}\}$  as projective coordinates for  $P(\sigma_1)$ . In these coordinates, the vertex cycles of  $\sigma_1$  are  $\gamma_1 = [1 : 0 : 0 : 0]$ ,  $\gamma_4 = [0 : 1 : 0 : 0]$ ,  $\gamma_2 = [0 : 0 : 1 : 0]$  and  $\alpha = [0 : 0 : 0 : 1]$ . By Proposition 4.2.5,  $P(\tau_1)$  and  $P(\sigma_1)$  intersect exactly in the polytope of the common subtrack  $\rho$  given by the equation  $w_5 = v_5 = 0$ . Note that any lamination carried by  $\rho$  satisfies  $w_k = v_k$  for all  $k$ . The vertex cycles of  $\rho$  are the curves  $\gamma_1, \gamma_2$  and  $\gamma_4$ . Define

$$\begin{aligned} \gamma_3 &= [1 : 0 : 0 : 1] = [w_1 : w_3 : w_4 : w_5] \in P(\tau), \\ \gamma_5 &= [0 : 0 : 1 : 1] = [v_1 : v_2 : v_4 : v_5] \in P(\sigma). \end{aligned}$$

The curves  $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$  form a pentagon  $\Gamma$  in  $\mathcal{C}(\Sigma_{0,5})$ , illustrated in Figure 4.9.

Observe that  $\eta(\Gamma)$  is contained in the frontier of  $P(\tau_1) \cup P(\sigma_1)$ . Explicitly:  $\gamma_1$  and  $\gamma_2$  are vertex cycles of  $\rho$ , so the image in  $\mathcal{PML}(\Sigma_{0,5})$  of the edge in  $\mathcal{C}(\Sigma_{0,5})$  connecting  $\gamma_1$  and  $\gamma_2$  is contained in  $\text{fr}(P(\rho)) \subset \text{fr}(P(\tau) \cup P(\sigma))$ ;  $\gamma_2, \gamma_3 \prec \tau$  lie in the face  $w_3 = 0$ , hence so is the image of the edge connecting them;  $\gamma_3, \gamma_4 \prec \tau$  lie in the face  $w_4 = 0$ ;  $\gamma_4, \gamma_5 \prec \sigma$  lie in the face  $v_1 = 0$ ;  $\gamma_5, \gamma_1 \prec \sigma$  lie in the face  $v_2 = 0$ . See Figure 4.10.  $\square$

### The decagon associated to a pentagon

Let  $\Gamma_{\mathcal{S}} = \{\gamma_1, \dots, \gamma_5\}$  be a pentagon on  $\Sigma_{0,5}$ , indexed so that  $\iota(\gamma_i, \gamma_{i+1}) = 0$ . We will describe a collection of curves  $\Delta_{\mathcal{S}}(\Gamma) = \{\delta_i : i \in \mathbb{Z}/10\} \subset \mathcal{S}(\Sigma_{0,5})$  with the



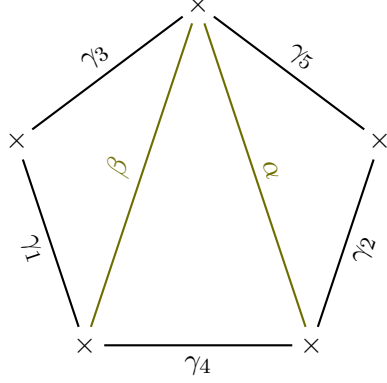


Figure 4.9: The curves (represented by arcs)  $\gamma_1, \dots, \gamma_5$  on  $\Sigma_{0,5}$  form a pentagon  $\Gamma$  in the curve graph.

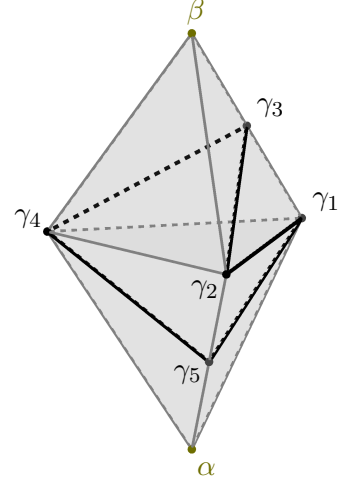


Figure 4.10: The pentagon  $\eta(\Gamma)$  in the frontier of the union of  $P(\tau_1)$  (top tetrahedron) and  $P(\sigma_1)$  (bottom).

property that  $\iota(\delta_i, \delta_j) = 0$  if and only if  $i \equiv j \pm 1$ .

As usual write  $H_\alpha$  for the half Dehn twist about the curve  $\alpha \in \mathcal{S}(\Sigma_{0,5})$ . Note that, if  $\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})$  intersect twice, then  $H_\alpha(\beta) = H_\beta^{-1}(\alpha)$ . For  $k \in \{1, 2, 3, 4, 5\}$ , define

$$\begin{aligned}\delta_{2k} &= H_{\gamma_k}^{-1}(\gamma_{k+2}), \\ \delta_{2k+1} &= H_{\gamma_k}^{-1}(\gamma_{k-2}).\end{aligned}$$

Note that  $\delta_{2k}$  and  $\delta_{2k+1}$  are disjoint, as  $\gamma_{k+2}$  and  $\gamma_{k-2}$  are. It takes one extra step to show that  $\delta_{2k-1}$  and  $\delta_{2k}$  are disjoint. Since the curve  $\gamma_{k-3} = \gamma_{k+2}$  intersects each of  $\gamma_{k-1}$  and  $\gamma_k$  twice, we find that  $\delta_{2k-1} = H_{\gamma_{k-1}}^{-1}(\gamma_{k+2}) = H_{\gamma_{k+2}}(\gamma_{k-1})$  and  $\delta_{2k} = H_{\gamma_k}^{-1}(\gamma_{k+2}) = H_{\gamma_{k+2}}(\gamma_k)$ . By definition  $\gamma_{k-1}$  and  $\gamma_k$  are disjoint, from which it follows that  $\iota(\delta_{2k-1}, \delta_{2k}) = 0$ . See Figures 4.11 and 4.12. Observe that  $\iota(\delta_i, \delta_j) = 0$  if and only if  $i \equiv j \pm 1$ . Consequently, the induced subgraph in  $\mathcal{C}(\Sigma_{0,5})$  on the vertex set  $\Delta_{\mathcal{S}}(\Gamma)$  is a decagon, denoted  $\Delta(\Gamma)$ .

**Definition 4.3.4** (Decagon).  $\Delta(\Gamma) \subset \mathcal{C}(\Sigma_{0,5})$  is called the *decagon associated to  $\Gamma$* .

Figure 4.13 depicts a pentagon and the associated decagon.

Suppose that  $\alpha_1, \alpha_2, \dots, \alpha_n$  is a sequence of curves on  $\Sigma_{0,5}$  with the property that  $\alpha_i$  and  $\alpha_{i+1}$  are connected by an edge  $e_i$  in  $\mathcal{C}(\Sigma_{0,5})$ , for every  $1 \leq i \leq n-1$ . We introduce the notation

$$[\alpha_1 \alpha_2 \dots \alpha_n] = \eta(e_1 \cup e_2 \cup \dots \cup e_{n-1}) \subset \mathcal{PML}(\Sigma_{0,5}).$$

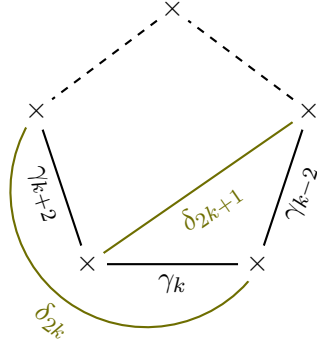


Figure 4.11: The curves (represented by arcs)  $\delta_{2k}$  and  $\delta_{2k+1}$  are disjoint.

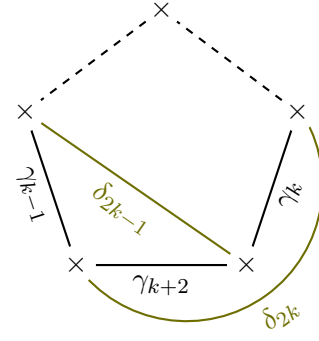


Figure 4.12: The curves (represented by arcs)  $\delta_{2k-1}$  and  $\delta_{2k}$  are disjoint.

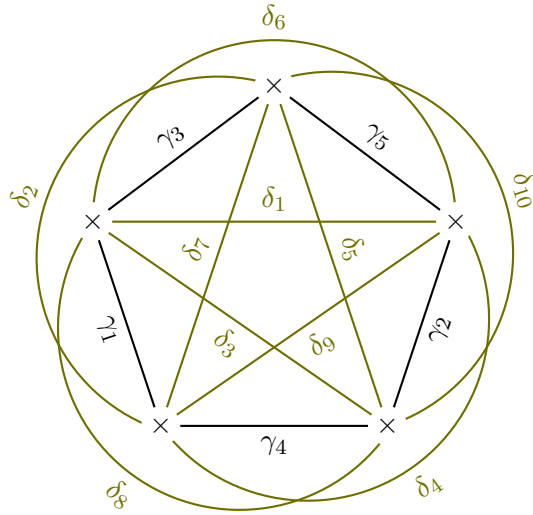


Figure 4.13: The decagon  $\Delta_{\mathcal{S}}(\Gamma) = \{\delta_i : i \in \mathbb{Z}/10\}$  corresponding to the pentagon  $\Gamma_{\mathcal{S}} = \{\gamma_i : i \in \mathbb{Z}/5\}$  of curves (represented by arcs) on  $\Sigma_{0,5}$ .

**Proposition 4.3.5.** *A pentagon and its decagon form a Hopf link in  $\mathcal{PML}(\Sigma_{0,5})$ .*

*Proof.* It suffices to prove the statement for the pentagon  $\Gamma_{\mathcal{S}} = \{\gamma_1, \dots, \gamma_5\}$  from Figure 4.13. Let  $\Delta = \Delta(\Gamma)$  be the decagon associated to  $\Gamma$ . In Proposition 4.3.3 we observed that the pentagon is an unknot. We will show that  $\eta(\Delta)$  bounds a PL disc that intersects  $\eta(\Gamma)$  in one point, and at this point  $\eta(\Gamma)$  crosses from one side of the disc to the other. This implies that  $\eta(\Gamma)$  and  $\eta(\Delta)$  form a Hopf link.

Consider the butterfly train tracks  $\tau_i, \sigma_i$  (Definition 4.2.1) and let  $r$  be the reflection so that  $r(\tau_1) = \sigma_1$ . As in the proof of Proposition 4.3.3, for  $1 \leq k \leq 6$  let  $w_k$  denote the weight on the small branch  $b_k$  of  $\tau_1$  and let  $v_k$  denote the weight on the small branch  $a_k$  of  $\sigma_1$  (Figures 4.7 and 4.8). Note that, in the notation of the proof of Proposition 4.3.3,  $\beta = \delta_7$  and  $\alpha = \delta_5$ . We first check which of the curves  $\delta_i \in \Delta$  are carried by  $\tau_1$  or  $\sigma_1$ .

$\tau_1$	$w_1$	$(w_2)$	$w_3$	$w_4$	$w_5$	$(w_6)$	$\sigma_1$	$v_1$	$v_2$	$(v_3)$	$v_4$	$v_5$	$(v_6)$
$\delta_1$	1	(1)	1	1	0	(2)	$\delta_1$	1	1	(1)	1	0	(2)
$\delta_2$	2	(1)	0	0	1	(0)	$\delta_3$	1	1	(1)	0	0	(2)
$\delta_3$	1	(1)	1	0	0	(2)	$\delta_5$	0	0	(1)	0	1	(0)
$\delta_7$	0	(1)	0	0	1	(0)	$\delta_9$	0	1	(1)	1	0	(2)
$\delta_9$	0	(1)	1	1	0	(2)	$\delta_{10}$	0	0	(1)	2	1	(0)

Note that  $\delta_9, \delta_{10}, \delta_1 \prec \sigma_1$  and  $\delta_1, \delta_2, \delta_3 \prec \tau_1$  hence  $[\delta_9 \delta_{10} \delta_1 \delta_2 \delta_3]$  is contained in  $P(\tau_1) \cup P(\sigma_1)$ .

We claim that the curves  $\delta_4, \delta_6$  and  $\delta_8$  are carried by neither  $\tau$  nor  $\sigma$ . We will show this using the butterfly train tracks. First of all,  $\delta_6$  is an interior point of  $P(\sigma_3) \cup P(\tau_3)$ . Proposition 4.2.5 implies that  $\delta_6$  cannot be contained in  $P(\tau_1)$  nor  $P(\sigma_1)$ . Secondly,  $\delta_4$  lies on the edge connecting  $\gamma_2$  and  $\gamma_4$  that is shared by  $\tau_3, \sigma_3, \tau_4$  and  $\sigma_4$ . By Proposition 4.2.5  $P(\tau_3) \cup P(\sigma_3) \cup P(\tau_4) \cup P(\sigma_4)$  is a neighbourhood of  $\delta_4$ , hence  $\delta_4$  cannot be carried by  $\tau_1$  nor  $\sigma_1$ . Thirdly,  $\delta_8 = r(\delta_4)$  and it follows that  $\delta_8$  is not carried by  $r(\tau_1) = \sigma_1$  nor  $r(\sigma_1) = \tau_1$ .

To avoid double indices, we will write  $D_i$  for the disc  $D_{\delta_i}$  in  $\mathcal{PML}(\Sigma_{0,5})$  (Definition 4.1.3). Note that

$$D_{2k} \cap \eta(\Gamma) = \{\gamma_{k+1}\} \quad \text{and} \quad D_k \cap \eta(\Gamma) = D_{k+5} \cap \eta(\Gamma),$$

where  $\eta$  is the map defined in Definition 4.1.1. Let  $D_i^+$  be the closed component of  $D_i - \eta(\Delta(\Gamma))$  that intersects  $\eta(\Gamma)$  and let  $D_i^-$  be the closed component that is disjoint from  $\eta(\Gamma)$ .

Write  $\text{conv}(Q; \tau)$  for the convex hull in  $P(\tau)$  of a collection of curves  $Q$  carried

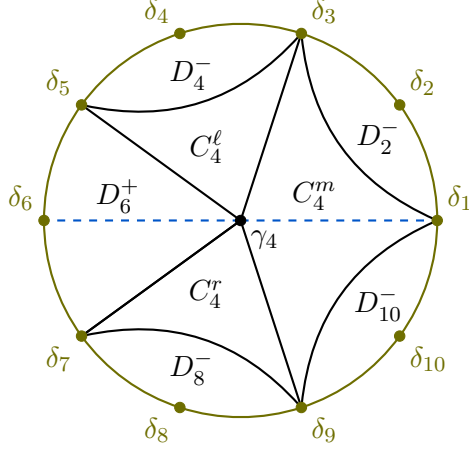


Figure 4.14: The piecewise linear disc  $U_4 \subset \mathcal{PML}(\Sigma_{0,5})$  bounded by the decagon.

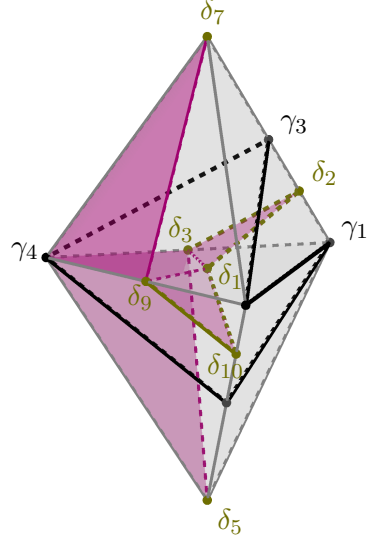


Figure 4.15: The intersection of the pentagon, the decagon and the disc  $U_4$  with  $P(\tau_1) \cup P(\sigma_1)$ .

by  $\tau$ . Observe that  $\text{conv}(\delta_1, \delta_2, \delta_3; \tau_1) = D_2^-$  and  $\text{conv}(\delta_9, \delta_{10}, \delta_1; \sigma_1) = D_{10}^-$ . Define  $C_4^\ell = \text{conv}(\delta_3, \delta_5, \gamma_4; \sigma_1)$ ,  $C_4^r = \text{conv}(\delta_7, \delta_9, \gamma_4; \tau_1)$  and  $C_4^m = \text{conv}(\delta_3, \delta_1, \delta_9, \gamma_4; \tau_1)$ .

We will prove the proposition by proving the following claim.

**Claim.** *The union of discs*

$$U_4 = D_2^- \cup D_4^- \cup D_6^+ \cup D_8^- \cup D_{10}^- \cup C_4^\ell \cup C_4^m \cup C_4^r$$

*is a disc in  $\mathcal{PML}(\Sigma_{0,5})$  with frontier  $\eta(\Delta)$ . The pentagon  $\eta(\Gamma)$  intersects the disc  $U_4$  only in  $\{\gamma_4\}$ , where it crosses from one side to the other.*

*Proof of claim.* We need to show that the eight discs that make up  $U_4$  have disjoint interiors and their frontiers assemble *correctly*, meaning ‘as indicated in Figure 4.14’. Observe that  $D_2^-, D_{10}^-, C_4^\ell, C_4^m$  and  $C_4^r$  are contained in  $P(\tau_1) \cup P(\sigma_1)$ . It is not hard to see that their union is a disc  $V_4$ , illustrated Figure 4.15.

We show that the remaining discs  $D_4^-, D_6^+$  and  $D_8^-$  meet  $P(\tau_1) \cup P(\sigma_1)$  only in their boundary, where they attach to  $V_4$  correctly. Observe that  $\delta_3, \delta_4, \delta_5 \prec \sigma_4$ . Let  $\sigma'_4 \prec \sigma_4$  and  $\sigma'_3 \prec \sigma_3$  be the train tracks depicted in Figure 4.16. Then  $P(\sigma'_4)$  is the triangle  $\text{conv}(\delta_3, \delta_4, \delta_5; \sigma_4)$ . In fact,  $P(\sigma'_4) = D_4^-$  and

$$P(\sigma'_4) \cap (P(\tau_1) \cup P(\sigma_1)) = \text{conv}(\delta_3, \delta_5; \sigma_1).$$

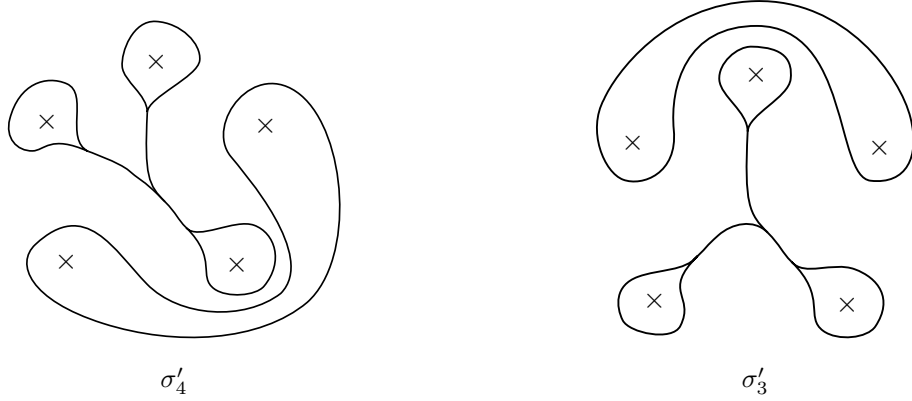


Figure 4.16: The train tracks  $\sigma'_4 \prec \sigma_4$  and  $\sigma'_3 \prec \sigma_3$ .

Similarly,  $\text{conv}(\delta_5, \delta_6, \gamma_4; \sigma_3) = P(\sigma'_3) \subset D_6$  and

$$P(\sigma'_3) \cap (P(\tau_1) \cup P(\sigma_1)) = \text{conv}(\delta_5, \gamma_4; \sigma_1).$$

Define  $\tau'_2 = r(\sigma'_4)$  and  $\tau'_3 = r(\sigma'_3)$ . Applying  $r$  to every observation of the previous paragraph, we find that  $\tau'_2 \prec \tau_2$ ,  $\tau'_3 \prec \tau_3$ ,  $\text{conv}(\delta_7, \delta_8, \delta_9; \tau_2) = P(\tau'_2) = D_8^-$ ,  $\text{conv}(\delta_6, \delta_7, \gamma_4; \tau_3) = P(\tau'_3) \subset D_6$  and

$$\begin{aligned} P(\tau'_2) \cap (P(\tau_1) \cup P(\sigma_1)) &= \text{conv}(\delta_7, \delta_9; \tau_1), \\ P(\tau'_3) \cap (P(\tau_1) \cup P(\sigma_1)) &= \text{conv}(\delta_7, \gamma_4; \tau_1). \end{aligned}$$

Furthermore, the triangles  $P(\sigma'_3)$  and  $P(\tau'_3)$  intersect in the edge connecting the disjoint curves  $\gamma_4$  and  $\delta_6$ , hence  $P(\sigma'_3) \cup P(\tau'_3) = D_6^+$ . This shows that  $D_4^-, D_6^+$  and  $D_8^-$  meet  $V_4$  correctly.

We now show that the discs  $D_4^-, D_6^+$  and  $D_8^-$  intersect each other correctly as well. To see that  $D_4^- \cap D_6^+ = \{\delta_5\}$ , note that the only lamination disjoint from both  $\delta_4$  and  $\delta_6$  is  $\delta_5$ . Similarly,  $D_6^+ \cap D_8^- = \{\delta_7\}$ . Since  $\delta_4$  and  $\delta_8$  fill the five-times punctured sphere, we find that  $D_4^- \cap D_8^- = \emptyset$ .

This proves that  $U_4$  is a disc with frontier  $\eta(\Delta)$ . The pentagon  $\eta(\Gamma)$  intersects  $U_4$  only in the point  $\gamma_4$ . Note that  $r(U_4) = U_4$ , swapping the sides of this disc. Since  $r(\Gamma) = \Gamma$ , we may conclude that  $\eta(\Gamma)$  crosses  $U_4$  from one side to the other.  $\square$

With the proof of the claim also the proof of the proposition is complete.  $\square$

## 4.4 The pentagon-decagon graph

This section further investigates the pentagon and its decagon. The main result is Theorem 4.4.6. Using this theorem, we are able to give an explicit description of a trefoil knot in the curve graph (Corollary 4.4.7).

Throughout the section, let  $\Gamma_{\mathcal{S}} = \{\gamma_m : m \in \mathbb{Z}/5\}$  be a pentagon of curves on  $\Sigma_{0,5}$  with associated decagon  $\Delta_{\mathcal{S}}(\Gamma) = \{\delta_n : n \in \mathbb{Z}/10\}$ , both indexed as in the previous section. We consider the following subgraph of the curve graph.

**Definition 4.4.1** (Pentagon-decagon graph). The *pentagon-decagon graph*  $\mathcal{G}_{\Gamma}$  is the induced subgraph of  $\mathcal{C}(\Sigma_{0,5})$  on the vertex set  $\Gamma_{\mathcal{S}} \cup \Delta_{\mathcal{S}}(\Gamma)$ .

Observe that  $\mathcal{G}_{\Gamma}$  contains the pentagon and the decagon as subgraphs. Every vertex of the pentagon is adjacent to two ‘opposite’ vertices of the decagon. Explicitly, in  $\mathcal{G}_{\Gamma}$  every  $\gamma_k$  is a vertex of valence 4 adjacent to  $\gamma_{k-1}, \gamma_{k+1}, \delta_{2k-2}$  and  $\delta_{2k+3}$ . Every  $\delta_k$  has valence 3 in  $\mathcal{G}_{\Gamma}$ .

We have the following corollary of the proof of Proposition 4.3.5.

**Lemma 4.4.2.** *For every  $0 \leq k \leq 5$  the 7-cycles*

$$[\gamma_k \delta_{2k-2} \delta_{2k-1} \delta_{2k} \delta_{2k+1} \delta_{2k+2} \delta_{2k+3} \gamma_k] \quad \text{and} \quad [\gamma_k \delta_{2k-2} \delta_{2k-3} \delta_{2k-4} \delta_{2k-5} \delta_{2k-6} \delta_{2k-7} \gamma_k]$$

*in  $\mathcal{PML}(\Sigma_{0,5})$  bound discs with disjoint interiors that intersect  $\mathcal{G}_{\Gamma}$  only in their frontier.*

*Proof.* Let  $\vartheta \in \text{MCG}(\Sigma_{0,5})$  be an order-5 mapping class that fixes  $\mathcal{G}_{\Gamma}$  setwise and maps the pair of 7-cycles for  $\gamma_k$  to the pair of 7-cycles for  $\gamma_{k-2}$ . Explicitly, in the situation of Figure 4.13 one could take  $\vartheta$  to be the mapping class that rotates the five-times punctured sphere  $2\pi/5$  anti-clockwise. Therefore, it suffices to prove the lemma for one fixed  $k$ .

Choose  $k = 4$ . Let  $U_4 \subset \mathcal{PML}(\Sigma_{0,5})$  be the disc bounded by  $\eta(\Delta(\Gamma))$  defined in the proof of Proposition 4.3.5 (see also Figure 4.14). The path  $[\delta_1 \gamma_4 \delta_6]$  is contained in  $U_4$  and cuts it into two discs. Hence  $U_4$  is the union of two discs that are bounded by the cycles

$$[\gamma_4 \delta_6 \delta_7 \delta_8 \delta_9 \delta_{10} \delta_1 \gamma_4] \quad \text{and} \quad [\gamma_4 \delta_6 \delta_5 \delta_4 \delta_3 \delta_2 \delta_1 \gamma_4],$$

and that have disjoint interiors. The proof of Proposition 4.3.5 implies that  $U_4$  intersects  $\mathcal{G}_{\Gamma}$  only in these two cycles. This proves the result for  $k = 4$ .  $\square$

## The standard embedding of $\mathcal{G}$ into $\mathbb{S}^3$

Write  $\mathcal{G}$  for the combinatorial graph underlying  $\mathcal{G}_\Gamma$ , forgetting the labelling of the vertices. For us, this graph has a preferred embedding into  $\mathbb{S}^3$ . We describe this standard embedding in two ways: first, using the quaternions; second, as a subcomplex of a simplicial structure on  $\mathbb{S}^3$ .

**Quaternion description of the standard embedding.** We identify  $\mathbb{R}^4$  with the *quaternions*

$$\{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

where  $1, i, j, k$  are the quaternion units. Recall that the quaternion multiplication  $i^2 = j^2 = k^2 = ijk = -1$  equips  $\mathbb{R}^4$  with a non-commutative group structure. Quaternions have been studied since the 1840s and, unsurprisingly, there are copious references for this material. See, for instance, [MorGS] for a recent account. An illuminating 3D-visualisation of quaternion multiplication is available at [San].

Identify  $\mathbb{S}^3$  with the unit sphere in  $\mathbb{R}^4$ . If  $A$  is a subset of  $\mathbb{R}^4$ , we write  $A^u$  for the set of unit vectors in  $A$ , that is,  $A^u = A \cap \mathbb{S}^3$ . Consider the planes  $H = \{a + bi : a, b \in \mathbb{R}\}$  and  $R = \{cj + dk : c, d \in \mathbb{R}\}$  in  $\mathbb{R}^4$ . Since  $H$  and  $R$  intersect only in the origin, the circles  $H^u$  and  $R^u$  are disjoint. We call these circles the *hub* and the *rim*, respectively. Define  $q = \cos(\frac{2\pi}{5}) + i \sin(\frac{2\pi}{5})$ . Consider the five half-planes

$$\begin{aligned} S_0 &= \{a + cj : a \geq 0, c \in \mathbb{R}\}, \\ S_m &= q^m S_0, \end{aligned}$$

where  $m \in \mathbb{Z}/5$ . Note that  $S_0^u$  is an arc connecting the points  $\pm j$  on the rim via the point 1 on the hub. Multiplication by  $q$  rotates the plane  $H$  anti-clockwise about the origin by an angle of  $\frac{2\pi}{5}$ . One can easily compute that  $qj = j \cos(\frac{2\pi}{5}) + k \sin(\frac{2\pi}{5})$  and  $qk = -j \sin(\frac{2\pi}{5}) + k \cos(\frac{2\pi}{5})$ . Therefore, multiplication by  $q$  rotates the plane  $R$  anti-clockwise about the origin by  $\frac{2\pi}{5}$  as well. It follows that  $S_m \cap S_n = \{0\}$  whenever  $m \not\equiv n$  in  $\mathbb{Z}/5$ . For any  $m \in \mathbb{Z}/5$ , the closed components of  $S_m^u - R$  are called *spokes*. Every spoke is an arc connecting the hub and the rim, and  $S_m^u$  is the union of two spokes.

The set

$$H^u \cup R^u \cup \bigcup_{m=1}^5 S_m^u$$

is the image of an embedding  $s: \mathcal{G} \hookrightarrow \mathbb{S}^3$  called the *standard embedding*. This

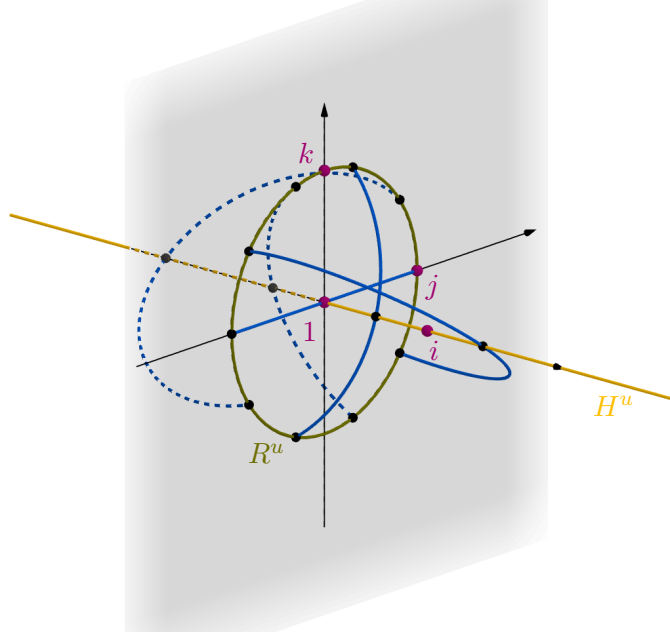


Figure 4.17: The standard embedding of  $\mathcal{G}$  in  $\mathbb{S}^3$  (where  $-1$  is at infinity). The shaded plane corresponds to  $i = 0$ . The dashed spokes satisfy  $i < 0$ , that is, they are ‘behind’ the shaded plane. Intersection points are marked with a dot.

embedded graph is illustrated in Figure 4.17. Here  $\mathbb{S}^3 \subset \mathbb{R}^4$  is stereographically projected to  $\mathbb{R}^3$  in such a way that  $-1$  is at infinity. Observe the analogy with a wheel with hub  $H^u$ , rim  $R^u$  and spokes  $\bigcup S_m^u$  in this projection. In practice, we say that an embedding  $\mathcal{G} \hookrightarrow \mathbb{S}^3$  is *standard* if it is isotopic to  $s$ , up to reversing orientation.

**Simplicial description of the standard embedding.** We now look at the standard embedding of  $\mathcal{G}$  from a viewpoint that is better suited to PL topology. Let  $\mathcal{X}$  be the 3-dimensional simplicial complex that is the simplicial join of the graphs  $\Gamma$  and  $\Delta(\Gamma)$ . Write  $\mathcal{X}_n$  for the  $n$ -skeleton of  $\mathcal{X}$ . Clearly, the graph  $\mathcal{G}_\Gamma$  is contained in  $\mathcal{X}_1$ . There is a homeomorphism  $h: \mathcal{X} \rightarrow \mathbb{S}^3 \subset \mathbb{R}^4$  with

$$\begin{aligned} h(\mathcal{X}_0) &= \{q^m x \mid m \in \mathbb{Z}/5, x \in \{1, j, -j\}\}, \\ h(\mathcal{X}_1) &= H^u \cup R^u \cup \{a + q^m c j \mid m \in \mathbb{Z}/5, a \geq 0, c \in \mathbb{R}\}^u, \\ h(\mathcal{X}_2) &= \{q^m(a + c j + d k) \mid m \in \mathbb{Z}/5, a \geq 0, c, d \in \mathbb{R}\}^u \\ &\quad \cup \{q^m(a + b i + c j) \mid m \in \mathbb{Z}/5, a, b, c \in \mathbb{R}\}^u, \end{aligned}$$



and  $h(\mathcal{G}_\Gamma) = s(\mathcal{G})$ . As a consequence, the embedding  $\mathcal{G}_\Gamma \subset \mathcal{X}$  is standard. Note that, if  $\vartheta$  denotes the anti-clockwise rotation of the five-times punctured sphere by  $2\pi/5$ , then  $\vartheta^2(x) = qh(x)$  for all  $x \in \mathcal{G}_\Gamma$ .

### The embedding of $\mathcal{G}_\Gamma$ in $\mathcal{PML}(\Sigma_{0,5})$ is standard

Our current goal is to prove Theorem 4.4.6. This theorem says that the embedding of  $\mathcal{G}_\Gamma$  in  $\mathcal{PML}(\Sigma_{0,5})$  is standard. The proof relies on a couple of topological lemmas, which we will prove first.

As above, let  $\mathcal{X}$  be the simplicial join of the graphs  $\Gamma$  and  $\Delta(\Gamma)$ . Let  $\mathcal{Y}$  be the 2-dimensional subcomplex of  $\mathcal{X}$  such that

- $\mathcal{Y}_0 = \mathcal{X}_0$ ,
- $\mathcal{Y}_1 = \mathcal{X}_1$ , and
- $\mathcal{Y}_2$  contains only the 2-simplices of  $\mathcal{X}$  with vertices  $\delta_n, \delta_{n+1}, \gamma_m$  for  $(n, m) \in \mathbb{Z}/10 \times \mathbb{Z}/5$ ,

where  $\mathcal{Y}_n$  denotes the  $n$ -skeleton of  $\mathcal{Y}$ . Clearly, the graph  $\Gamma$  is contained in  $\mathcal{Y}_1$ .

**Lemma 4.4.3.** *Suppose that  $\mathbb{S}^3$  is equipped with a PL structure. A PL embedding  $f: \mathcal{Y} \hookrightarrow \mathbb{S}^3$  extends to a homeomorphism  $\mathcal{X} \rightarrow \mathbb{S}^3$  if and only if  $f(\Gamma)$  is the unknot.*

*Proof.* It is easy to find a disc in  $\mathcal{X}_2$  bounded by  $\Gamma$ , from which the ‘only if’ statement follows.

We proceed with the ‘if’ statement. For  $m \in \mathbb{Z}/5$ , the union of all closed 2-simplices of  $\mathcal{Y}$  incident to  $\gamma_m$  is a disc  $V_m$  with boundary  $\Delta(\Gamma)$ . By the Schönflies theorem (see, for instance [Moi, Theorem 17.12]), the PL embedded 2-sphere  $f(V_m \cup V_{m+1})$  bounds a 3-ball on either side. Let  $e_m$  denote the edge of  $\Gamma$  with endpoints  $\gamma_m$  and  $\gamma_{m+1}$ . Let  $B_m \subset \mathbb{S}^3$  be the ball bounded by  $f(V_m \cup V_{m+1})$  that contains the embedded edge  $f(e_m)$ .

**Claim.**  $f(\mathcal{Y}) \cap \text{int } B_m = f(e_m)$ .

*Proof of claim.* Observe that  $\mathcal{Y} = \bigcup_{n \in \mathbb{Z}/5} V_n \cup e_n$ , so it suffices to show that  $\text{int } B_m$  is disjoint from every  $f(V_n)$  and every  $f(e_n)$  with  $n \neq m$ .

Suppose that  $f(V_n)$  intersects  $\text{int}(B_m)$  for some  $n \neq m, m+1$ . Again applying the Schönflies theorem, we find that  $f(V_m \cup V_n)$  and  $f(V_{m+1} \cup V_n)$  bound balls in  $B_m$  with disjoint interiors. In particular,  $f(\gamma_m)$  and  $f(\gamma_{m+1})$  are contained in distinct connected components of  $B_m - f(V_n)$ . This implies that  $f(e_m)$  intersects  $f(V_n)$ ,

contradicting the assumption that  $f$  is injective. We conclude that the interior of  $B_m$  is disjoint from every  $f(V_n)$ .

Now suppose that  $f(e_n)$  intersects the interior of  $B_m$ . In that case  $f(e_n) \subset B_m$ , hence  $f(\gamma_n), f(\gamma_{n+1}) \in B_m$ . The previous paragraph shows that  $f(\gamma_n)$  and  $f(\gamma_{n+1})$  lie in the frontier of  $B_m$ , hence  $n = m$ .  $\square$

We will argue that  $f(e_m)$  is an ‘unknotted’ arc in  $B_m$ . Recall that a properly embedded arc in a 3-ball  $B$  is called *unknotted* if its image under the quotient map  $B \rightarrow \mathbb{S}^3$  that identifies all points on  $\text{fr } B$  is the unknot. If the connected sum of a number of knots is the unknot, then each knot in the sum is the unknot [Sch]. Since  $f(\Gamma)$  is the connected sum of the quotient knots of the arcs  $f(e_1), f(e_2), \dots, f(e_5)$ , every  $f(e_m)$  must be an unknotted arc in  $B_m$ .

Extend  $f$  over the 3-simplices of  $\mathcal{X}$  as follows. Let  $A_m \subset \mathcal{X}$  be the union of all closed 3-simplices incident to  $e_m$ . Since  $f(e_m)$  is unknotted, there exists a homeomorphism  $A_m \rightarrow B_m$  that agrees with  $f$  on  $V_m \cup V_{m+1} \cup e_m$ . Doing this for every  $m \in \mathbb{Z}/5$  we obtain a homeomorphism  $\mathcal{X} \rightarrow \mathbb{S}^3$  extending  $f$ .  $\square$

We also need the following general fact about embeddings of CW complexes.

**Lemma 4.4.4.** *Let  $X$  be a regular finite 2-dimensional CW complex. Let  $X_n$  denote the  $n$ -skeleton of  $X$ . Assume that the intersection of any two 2-cells of  $X$  is contractible. Suppose we are given an embedding  $f: X_1 \rightarrow \mathbb{S}^3$  such that for any 2-cell  $C$  of  $X$ , there exists a disc  $D$  in  $\mathbb{S}^3$  with*

$$D \cap f(X_1) = \text{fr } D = f(\text{fr } C).$$

*Then  $f$  extends to an embedding of  $X$  into  $\mathbb{S}^3$ .*

*Proof.* We use a surgery argument to prove this lemma. Let  $c_1, c_2, \dots, c_m$  be the frontier curves of the 2-cells of  $X$ . For every  $1 \leq i \leq m$  choose an embedded disc  $D_i$  in  $\mathbb{S}^3$  such that  $D_i \cap f(X_1) = \text{fr } D_i = f(c_i)$ . We may assume that these discs are in *general position*, that is, every disc meets  $f(X_1)$  only in its boundary, any two discs intersect transversely and no point is interior to three or more discs.

For any  $1 \leq i \leq m$  the set

$$\bigsqcup_{j \neq i, 1 \leq j \leq m} D_j \cap \text{int } D_i$$

is a finite union of properly embedded curves and arcs in  $D_i$ . We start by removing the intersection curves. Let  $c$  be a curve in  $D_j \cap \text{int } D_i$  that is innermost for  $D_i$ .

First swap the discs that  $c$  bounds in  $D_i$  and  $D_j$ , then perform a small isotopy so that the redefined discs  $D_i$  and  $D_j$  are in general position and the total number of intersection curves has reduced by 1. Induct on the number of intersection curves to remove all intersection curves.

Assume that interiors of the discs  $D_1, D_2, \dots, D_m$  only intersect in arcs. By assumption  $c_{ij} = c_i \cap c_j$  is contractible, so  $c_{ij}$  is homeomorphic to  $\emptyset, \{*\}$  or  $I$  for any  $1 \leq i, j \leq m$ . For every component  $a$  of  $\text{int}(D_i) \cap D_j$  there exists a unique embedded arc  $\mathbf{a} \subset f(c_{ij})$  so that  $\bar{a} = a \cup \mathbf{a}$  is an embedded curve in  $D_i \cap D_j$ . We will describe a notion of ‘innermost arc’ for  $D_i$ . For any two intersection arcs  $a, b \subset \text{int } D_i$ , write  $a > b$  if there exist (not necessarily distinct)  $j, k \neq i$  such that  $a \subset D_j$ ,  $b \subset D_k$  and

$$\mathbf{b} \subset \mathbf{a} \subset f(c_{ij}).$$

This defines a partial order on the collection of intersection arcs in  $\text{int } D_i$ . Say that an intersection arc is *innermost for  $D_i$*  if it is a minimal element for this order. Observe that if  $a$  is an innermost arc for  $D_i$ , then  $\bar{a}$  bounds a disc in  $D_i$  that does not contain any intersection arcs in its interior. For, if not, then there would be an arc  $b \in D_k$  so that  $\mathbf{a} \cap \mathbf{b} \subset c_{ij} \cap c_k$  has two components. If  $k = j$  this implies that  $c_{ij} \cong \mathbb{S}^1$ . If  $k \neq j$  this implies that  $c_{jk}$  is disconnected. In both cases we obtain a contradiction, since every  $c_{ij}$  is contractible.

Let  $a \subset D_j \cap \text{int } D_i$  be an intersection arc that is innermost for  $D_i$ . Swap the discs that the curve  $\bar{a}$  bounds in  $D_i$  and  $D_j$ . Next, perform a small isotopy to obtain two redefined discs  $D_i$  and  $D_j$  that are in general position and so that the total number of intersection arcs reduces by 1. We remove all intersection arcs inductively. It follows that we can choose the discs  $D_i$  to have disjoint interiors.

Mapping the 2-cell  $C_i$  with frontier curve  $c_i$  to the disc  $D_i$  extends  $f$  to an embedding  $f: X \rightarrow \mathbb{S}^3$ .  $\square$

*Remark 4.4.5.* Lemma 4.4.4 is still true in the PL category: in the situation of the lemma, if  $X$  and  $\mathbb{S}^3$  can be equipped with PL structures so that the embedding  $f: X_1 \rightarrow \mathbb{S}^3$  is piecewise linear, then  $f$  can be extended to a PL embedding of  $X$ .

**Theorem 4.4.6.** *The embedding  $\eta: \mathcal{G}_\Gamma \hookrightarrow \mathcal{PML}(\Sigma_{0,5})$  is standard.*

*Proof.* Let  $\mathcal{X}$  and  $\mathcal{Y}$  be the simplicial complexes defined earlier in this section. Let  $Y$  be the 2-dimensional CW complex defined by the properties  $Y_0 = \mathcal{Y}_0$ ,  $Y_1 = \mathcal{G}_\Gamma \subset \mathcal{Y}_1$  and  $Y_2 = \mathcal{Y}_2$ . Note that  $Y$  can be subdivided to  $\mathcal{Y}$ , giving it a PL structure.

Let  $\eta: Y_1 \hookrightarrow \mathcal{PML}(\Sigma_{0,5})$  be the PL embedding that is the restriction to  $\mathcal{G}_\Gamma$  of the map  $\eta$  from Definition 4.1.1. Lemma 4.4.2 implies that if  $c$  is the frontier

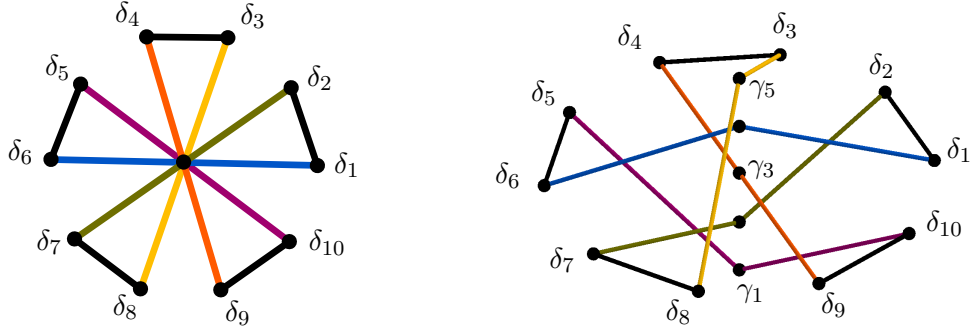


Figure 4.18: Two projections of the knot  $K$  inside  $\mathcal{PML}(\Sigma_{0,5})$ .

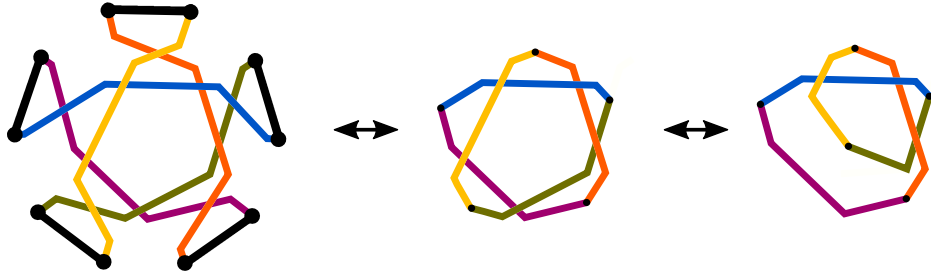


Figure 4.19: Reidemeister moves show that  $K$  is a trefoil knot.

of a 2-cell of  $Y$  then  $\eta(c)$  bounds a disc in  $\mathcal{PML}(\Sigma_{0,5})$  whose interior is disjoint from  $\eta(Y_1)$ . Furthermore, the intersection of any two 2-cells of  $Y$  is contractible. By Lemma 4.4.4 the map  $\eta$  extends to an embedding  $\tilde{\eta}: Y \rightarrow \mathcal{PML}(\Sigma_{0,5})$ . We may assume that  $\tilde{\eta}$  is piecewise linear (Remark 4.4.5). By Proposition 4.3.3, the pentagon is unknotted in  $\mathcal{PML}(\Sigma_{0,5})$ . Now Lemma 4.4.3 says that  $\tilde{\eta}$  extends to a homeomorphism  $\tilde{\eta}: \mathcal{X} \rightarrow \mathcal{PML}(\Sigma_{0,5})$ . This proves that  $\eta: \mathcal{G} \hookrightarrow \mathcal{PML}(\Sigma_{0,5})$  is standard.  $\square$

**Corollary 4.4.7.** *The loop*

$$K = [\delta_1 \gamma_4 \delta_6 \delta_5 \gamma_1 \delta_{10} \delta_9 \gamma_3 \delta_4 \delta_3 \gamma_5 \delta_8 \delta_7 \gamma_2 \delta_2 \delta_1]$$

*in  $\mathcal{PML}(\Sigma_{0,5})$  is a trefoil knot.*

*Proof.* This can be concluded from Theorem 4.4.6 after inspection of the standard embedding of the graph  $\mathcal{G}$ . Figure 4.18 (left) shows a non-regular projection and (right) shows a regular projection of this loop. A sequence of Reidemeister moves brings  $K$  into a standard projection of a trefoil knot, see Figure 4.19.  $\square$

The non-regular projection depicted in Figure 4.18 is also known as a ‘petal projection’ of the trefoil knot. See, for instance, [EvHLN].

*Remark 4.4.8.* Corollary 4.4.7 displays neatly that  $\mathcal{C}(\Sigma_{0,5})$  is mapped into  $\mathcal{PML}(\Sigma_{0,5})$  with a particular chirality. This uses the fact that the trefoil knot is not isotopic to its mirror image, which was first observed by Dehn [Deh1].

## Chapter 5

# Topologies on boundary laminations

This chapter introduces and compares several topologies on the set of boundary laminations on  $\Sigma_{0,5}$ , coming from various notions of convergence.

In Section 5.1 we define convergence in subsurface projection and prove that this is equivalent to superconvergence (Proposition 5.1.7). As a consequence, the associated topologies on  $\mathcal{BL}$  are the same (Proposition 5.1.9). We use these results in Section 5.2 to describe a neighbourhood base for  $(\mathcal{BL}, \mathbf{T}_s)$ .

Section 5.3 recalls a number of known results, notably Theorem 5.3.4 due to Brock and Masur [BroM, Theorem 5]. From these results we deduce that  $(\mathcal{BL}, \mathbf{T}_s)$  is homeomorphic to the Gromov boundary of  $\mathcal{P}(\Sigma_{0,5})$  (Corollary 5.3.5).

Section 5.4 compares the topology of superconvergence on  $\mathcal{BL}$  to the measure topology. Proposition 5.4.4 summarises the key observations of this section.

### 5.1 Convergence in subsurface projection

The goal of this section is to prove Proposition 5.1.7. Informally, this says that on the five-times punctured sphere superconvergence (Definition 3.2.4) and convergence in subsurface projection (Definition 5.1.1) are equivalent.

Brock and Masur used subsurface projections to give a notion of convergence for sequences of boundary laminations [BroM, §4]. We extend this to all measurable laminations on  $\Sigma_{0,5}$ .

**Definition 5.1.1** (Convergence in subsurface projection). Let  $(\lambda_i)_{i \in \mathbb{N}}$  be a sequence of measurable laminations on  $\Sigma_{0,5}$ . We say that  $(\lambda_i)$  *converges in subsurface projection*

to  $\lambda \in \mathcal{BL}$  if, taking  $W = \text{Supp}(\lambda)$ , we have that for every  $\mu_i \in \pi_W(\lambda_i)$  the sequence  $(\mu_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ .

To build up to the proof of Proposition 5.1.7 we have to do some work. We give a notion of superconvergence for sequences of curves and arcs (Definition 5.1.3 and show this is well defined (Lemma 5.1.5). Roughly, Lemma 5.1.6 says that replacing arcs or curves by disjoint ones does not affect superconvergence to an ending lamination.

Throughout the section let  $\Sigma = \Sigma_{g,n}$  be a surface without boundary, endowed with a fixed hyperbolic structure.

We defined superconvergence for sequences of laminations on  $\Sigma$  in Definition 3.2.4. Gabai gives the following equivalent characterisation [Gab1, Remark 3.3]. We will switch between the two characterisations without explicit mention.

**Lemma 5.1.2** (Gabai). *For every  $i \in \mathbb{N}$  let  $\lambda_i \in \mathcal{L}(\Sigma)$ . The sequence  $(\lambda_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda \in \mathcal{L}(\Sigma)$  if and only if  $\lambda$  is a sublamination of the limit of any convergent subsequence of  $(\lambda_i)_{i \in \mathbb{N}}$  in the Hausdorff topology on  $\mathcal{L}(\Sigma)$ .*

Superconvergence can be defined more generally for any sequence  $(Y_i)_{i \in \mathbb{N}}$  of subsets of a space  $X$ . Say that  $(Y_i)_{i \in \mathbb{N}}$  *superconverges* to  $Y \subset X$ , denoted

$$Y_i \xrightarrow{s} Y,$$

if for any  $y \in Y$  there exist  $y_i \in Y_i$  such that  $y_i \rightarrow y$  in  $X$ . We use this to define superconvergence for sequences of arcs.

**Definition 5.1.3** (Superconvergence of curves and arcs). Let  $W \subset \Sigma$  be a domain with (possibly empty) geodesic boundary. If  $\lambda \in \mathcal{EL}(W)$  and  $\alpha_i \in \mathcal{S}(W) \cup \mathcal{A}(W)$  for any  $i \in \mathbb{N}$ , we say that  $(\alpha_i)_{i \in \mathbb{N}}$  *superconverges* to  $\lambda$  if  $a_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$  for some geodesic representative  $a_i$  of  $\alpha_i$ .

We need to show that Definition 5.1.3 is well defined, that is, does not depend on the choice of geodesic representatives. Every curve has a unique geodesic representative (Example 3.2.1). Arcs do not have a unique geodesic representative (since we do not require isotopic arcs to have the same endpoints). However, arcs do have a unique *shortest* geodesic representative.

*Remark 5.1.4.* Let  $W$  be a proper domain in  $\Sigma$  with geodesic boundary. For any proper arc  $\mathbf{a}: I \rightarrow W$  the unique geodesic of minimal length  $\mathbf{b}: I \rightarrow W$  homotopic to  $\mathbf{a}$  is embedded. To prove this, let  $\widetilde{W} \subset \mathbb{H}^2$  be the universal cover of  $W$ . The homotopy that takes  $\mathbf{a}$  to  $\mathbf{b}$  lifts to an homotopy  $\widetilde{W} \times I \rightarrow \widetilde{W}$ . If  $\mathbf{b}$  is not embedded,

then there exist two intersecting arcs  $\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2$  lifting  $\mathbf{b}$ . The geodesic arcs  $\tilde{\mathbf{b}}_1$  and  $\tilde{\mathbf{b}}_2$  connect distinct geodesics that lift components of  $\partial W$ . The corresponding lifts  $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2$  of  $\mathbf{a}$  connect the same pairs of geodesics, so  $\tilde{\mathbf{a}}_1 \cap \tilde{\mathbf{a}}_2 \neq \emptyset$ . This contradicts that  $\mathbf{a}$  is embedded.

Since isotopy and homotopy are the same for arcs [FarM, §1.2.7], we deduce that every element of  $\mathcal{A}(W)$  has a unique shortest geodesic representative.

Recall that  $\text{PT}\Sigma$  denotes the unit tangent bundle of  $\Sigma$  (Remark 3.2.3). For a subset  $\lambda$  of  $\Sigma$  that is foliated by geodesics, we write  $\lambda^{\text{PT}}$  for the corresponding subset of  $\text{PT}(\Sigma)$ .

The following lemma proves that Definition 5.1.3 is well defined.

**Lemma 5.1.5.** *Suppose that  $W \subset \Sigma$  is a proper domain with geodesic boundary and let  $\lambda \in \mathcal{EL}(W)$ . For every  $i \in \mathbb{N}$  let  $\mathbf{a}_i$  be a geodesic arc with endpoints on  $\partial W$  and let  $\mathbf{b}_i$  be the shortest geodesic arc isotopic to  $\mathbf{a}_i$ . Then*

$$\mathbf{a}_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}} \iff \mathbf{b}_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}.$$

*Proof.* We will prove three claims which then combine to prove the lemma. Let  $W \subset \Sigma$  be a proper domain with geodesic boundary. Let  $\mathbf{b}$  be a geodesic arc on  $W$  that is shortest in its isotopy class – in other words,  $\mathbf{b}$  meets  $\partial W$  perpendicularly. Write  $\text{length}(\mathbf{b})$  for the hyperbolic length of  $\mathbf{b}$ .

**Claim.** *There exists a constant  $K_1 \geq 0$  depending on  $\text{length}(\mathbf{b})$  and with  $K_1 \rightarrow 0$  as  $\text{length}(\mathbf{b}) \rightarrow \infty$  such that the following is true. For every geodesic arc  $\mathbf{a}$  isotopic to  $\mathbf{b}$ , there exist  $x \in \mathbf{a}^{\text{PT}}$  and  $y \in \mathbf{b}^{\text{PT}}$  with  $\text{d}_{\text{PT}}(x, y) \leq K_1$ .*

*Proof of claim.* To prove this claim, let  $\tilde{\mathbf{b}}: [0, t] \rightarrow \tilde{W} \subset \mathbb{H}^2$  be a lift of  $\mathbf{b}$  to the universal cover, connecting geodesics  $d_0$  and  $d_1$  that lift boundary components of  $W$ . Let  $G$  be the set of geodesic arcs connecting a point on  $d_0$  to a point on  $d_1$ . Define

$$K_1 = \sup_{g \in G} \inf_{x \in g, y \in \tilde{\mathbf{b}}} \text{d}_{\text{PT}}(x, y)$$

It is an elementary property of hyperbolic geometry that  $K_1$  is finite. Furthermore,  $K_1$  depends continuously on the length of  $\mathbf{b}$  and  $K_1 \rightarrow 0$  as  $\text{length}(\mathbf{b}) \rightarrow \infty$ . Lift the isotopy between  $\mathbf{b}$  and  $\mathbf{a}$  to find a lift  $\tilde{\mathbf{a}}$  of  $\mathbf{a}$  that connects  $d_0$  and  $d_1$ . By definition of  $K_1$  there exist  $\tilde{x} \in \tilde{\mathbf{a}}^{\text{PT}}$  and  $\tilde{y} \in \tilde{\mathbf{b}}^{\text{PT}}$  such that  $\text{d}_{\text{PT}}(\tilde{x}, \tilde{y}) \leq K_1$ . The projections of  $\tilde{x}$  and  $\tilde{y}$  to  $W$  are no further apart, giving  $x \in \mathbf{a}^{\text{PT}}$  and  $y \in \mathbf{b}^{\text{PT}}$  with  $\text{d}_{\text{PT}}(x, y) \leq K_1$ .  $\square$

Suppose that  $\lambda, (\mathbf{a}_i)_{i \in \mathbb{N}}$  and  $(\mathbf{b}_i)_{i \in \mathbb{N}}$  are as in the statement of the lemma.



**Claim.** For all  $i \in \mathbb{N}$  let  $x_i \in \mathbf{a}_i^{\text{PT}}$ . Then  $\mathbf{a}_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$  if and only if for every accumulation point  $x \in \text{PT } W$  of  $(x_i)_{i \in \mathbb{N}}$  the geodesic extending  $x$  over  $W$  is dense in  $\lambda$ .

*Proof of claim.* We start with the ‘only if’ part. Suppose that  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda \in \mathcal{EL}(W)$ . Let  $x$  be an accumulation point of  $(x_i)_{i \in \mathbb{N}}$  and let  $g$  be the geodesic on  $W$  obtained by extending  $x$ . Observe that  $g$  cannot intersect  $\lambda$  transversely. Since  $\lambda$  is filling,  $g$  is not a simple closed geodesic. So  $g$  is a geodesic ray or line, and minimality of  $\lambda$  gives that  $g$  is dense in  $\lambda$ .

For the ‘if’ part, assume that for every accumulation point  $x$  of  $(x_i)_{i \in \mathbb{N}}$  the geodesic extending  $x$  over  $W$  is dense in  $\lambda$ . For any subsequence of  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  take a further subsequence  $(\mathbf{a}_{i_j})_{j \in \mathbb{N}}$  such that the corresponding subsequence  $(x_{i_j})_{j \in \mathbb{N}}$  is convergent. Let  $x$  be the limit point of  $(x_{i_j})_{j \in \mathbb{N}}$  and let  $g$  be the geodesic extending  $x$ . Then  $(\mathbf{a}_{i_j})_{j \in \mathbb{N}}$  superconverges to the geodesic  $g$  extending  $x$  and therefore to  $\lambda$ . So every subsequence of  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  has a subsequence that superconverges to  $\lambda$ , from which it follows that  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$  itself.  $\square$

**Claim.** If  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  or  $(\mathbf{b}_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda \in \mathcal{EL}(W)$  then  $\text{length}(\mathbf{b}_i) \rightarrow \infty$ .

*Proof of claim.* Suppose that there is some  $K > 0$  such that  $\text{length}(\mathbf{b}_i) \leq K$  for all  $i \in \mathbb{N}$ . There are only finitely many isotopy classes of arcs whose shortest representative has length at most  $K$ , hence up to passing to a subsequence  $(\mathbf{b}_i)_{i \in \mathbb{N}}$  is constant. It follows that  $(\mathbf{b}_i)_{i \in \mathbb{N}}$  does not superconverge to  $\lambda$ .

To see that  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  cannot superconverge to  $\lambda$  either, let  $\tilde{\mathbf{b}}$  be a lift of  $\mathbf{b} = \mathbf{b}_i$  to  $\tilde{W} \subset \mathbb{H}^2$  connecting lifted boundary curves  $d_0$  and  $d_1$ . Lift the isotopy that takes  $\mathbf{b}_i$  to  $\mathbf{a}_i$  to obtain an arc  $\tilde{\mathbf{a}}_i$  with endpoints on  $d_0$  and  $d_1$  that lifts  $\mathbf{a}_i$ . Since  $\lambda$  is filling, it has a leaf  $\ell$  that intersects  $\mathbf{b}$  transversely. Let  $\tilde{\ell}$  be a lift of  $\ell$  that intersects  $\tilde{\mathbf{b}}$ . Since  $\lambda$  is a closed subset of the interior of  $W$ , the endpoints of  $\tilde{\ell}$  are distinct from the endpoints of  $d_0$  and  $d_1$ . However,  $\tilde{\mathbf{a}}_i$  can only accumulate on geodesics connecting (endpoints of)  $d_0$  and  $d_1$ . It follows that any accumulation point of  $(\tilde{\mathbf{a}}_i)_{i \in \mathbb{N}}$  intersects  $\tilde{\ell}$  transversely. Therefore  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  does not superconverge to  $\lambda$ .  $\square$

The lemma now follows by combination of the three claims. Suppose that  $(\mathbf{a}_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . The third claim gives that  $\text{length}(\mathbf{b}_i) \rightarrow \infty$ . Using the first claim, choose  $x_i \in \mathbf{a}_i^{\text{PT}}$  and  $y_i \in \mathbf{b}_i^{\text{PT}}$  such that  $\lim_i d_{\text{PT}}(x_i, y_i) = 0$ . By the second claim, for any accumulation point  $x \in \text{PT } W$  of  $(x_i)_{i \in \mathbb{N}}$  the geodesic extending  $x$  over  $W$  is dense in  $\lambda$ . Accumulation points of  $(x_i)_{i \in \mathbb{N}}$  and  $(y_i)_{i \in \mathbb{N}}$  are the same, so for any accumulation point  $y \in \text{PT } W$  of  $(y_i)_{i \in \mathbb{N}}$  the geodesic extending  $y$  over  $W$  is

dense in  $\lambda$ . It now follows from the second claim that  $(\mathbf{b}_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . The opposite implication is similar.  $\square$

Informally, Lemma 5.1.6 says that sequences of disjoint laminations, arcs or curves superconverge to the same set. A similar result can be found in Wickens's thesis [Wic, Lemma 2.1.10].

**Lemma 5.1.6.** *Let  $W \subset \Sigma$  be a domain with (possibly empty) geodesic boundary and suppose that  $\lambda \in \mathcal{EL}(W)$ .*

- (i) *If  $\lambda_i, \mu_i \in \mathcal{L}(W)$  with  $\iota(\lambda_i, \mu_i) = 0$  for every  $i \in \mathbb{N}$  and  $\lambda_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$ , then  $\mu_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$ .*
- (ii) *Assume that  $W$  has non-empty geodesic boundary. Let  $\alpha_i, \beta_i \in \mathcal{S}(W) \cup \mathcal{A}(W)$  such that  $\iota(\alpha_i, \beta_i) = 0$  for all  $i \in \mathbb{N}$ . If  $(\alpha_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$  then  $(\beta_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$  as well.*

*Proof.* For part (i), suppose that  $\lambda_i$  and  $\mu_i$  are disjoint laminations on  $W$ . Suppose that  $(\lambda_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . Note that  $\lambda_i \cup \mu_i$  is a lamination on  $W$  for any  $i \in \mathbb{N}$  and the sequence  $(\lambda_i \cup \mu_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . Let  $\Lambda_\mu$  (respectively  $\Lambda$ ) be the subset of  $\mathcal{L}(W)$  consisting of all accumulation points of  $(\mu_i)_{i \in \mathbb{N}}$  (respectively  $(\lambda_i \cup \mu_i)_{i \in \mathbb{N}}$ ) in the Hausdorff topology. For every  $\kappa \in \Lambda_\mu$  there exists some  $\kappa' \in \Lambda$  with  $\kappa \subset \kappa'$ . But also  $\lambda \subset \kappa'$  and  $\lambda$  is ending, so  $\lambda$  must be contained in  $\kappa$ . This shows that  $(\mu_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ .

For part (ii), suppose that  $W$  is a proper domain with geodesic boundary. Let  $\alpha_i, \beta_i$  as in the statement and suppose that  $(\alpha_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . Let  $\mathbb{W}$  be the *double* of  $W$ , that is, the hyperbolic surface obtained by gluing two copies of  $W$  along their boundary using the identity map. Let  $\varphi: \mathbb{W} \rightarrow W$  denote the 2-fold branched cover that is the identity on each copy of  $W$ . The inclusion of one copy  $W \subset \mathbb{W}$  defines an inclusion  $d: \mathcal{S}(W) \hookrightarrow \mathcal{S}(\mathbb{W})$ . The preimage under  $\varphi$  of an arc on  $W$  is a curve on  $\mathbb{W}$ , defining an inclusion  $d: \mathcal{A}(W) \hookrightarrow \mathcal{S}(\mathbb{W})$ . The map  $d$  preserves disjointness of curves and arcs. Note that a sequence of curves and arcs on  $W$  superconverges to  $\lambda \in \mathcal{EL}(W)$  if and only if its image under  $d$  superconverges to  $\lambda \in \mathcal{EL}(W) \subset \mathcal{UML}(\mathbb{W})$ . Therefore, to prove the lemma it is equivalent to show that  $(d(\beta_i))_{i \in \mathbb{N}}$  superconverges to  $\lambda$  if  $(d(\alpha_i))_{i \in \mathbb{N}}$  superconverges to  $\lambda$ .

Suppose that  $(d(\alpha_i))_{i \in \mathbb{N}}$  superconverges to  $\lambda \in \mathcal{EL}(W)$ . For every  $i \in \mathbb{N}$  let  $a_i, b_i$  be the simple closed geodesic on  $\mathbb{W}$  representing  $d(\alpha_i), d(\beta_i)$  respectively. Since  $a_i$  and  $b_i$  are disjoint,  $a_i \cup b_i$  is a lamination on  $\mathbb{W}$  and  $(a_i \cup b_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . Let  $\Lambda_b$  (respectively  $\Lambda$ ) be the subset of  $\mathcal{L}(\mathbb{W})$  consisting of all accumulation points of  $(b_i)_{i \in \mathbb{N}}$  (respectively  $(a_i \cup b_i)_{i \in \mathbb{N}}$ ) in the Hausdorff topology. Every  $\kappa \in \Lambda_b$

intersects  $\partial W$  perpendicularly and  $|\kappa \cap \partial W| \leq 2$ . As a consequence,  $\kappa$  contains a minimal lamination on  $W$  as a sublamination. Choose  $\kappa' \in \Lambda$  such that  $\kappa \subset \kappa'$ . Then  $\kappa'$  contains both  $\kappa$  and  $\lambda$  as sublamination. Since  $\kappa$  contains a minimal lamination on  $W$ , we conclude that  $\lambda \subset \kappa$ . Hence  $(d(\beta_i))_{i \in \mathbb{N}}$  superconverges to  $d(\beta)$ .  $\square$

We are now prepared to prove the most important result of this section, Proposition 5.1.7.

**Proposition 5.1.7.** *Let  $\lambda$  be a boundary lamination and let  $(\lambda_i)_{i \in \mathbb{N}}$  be a sequence of measurable laminations on  $\Sigma_{0,5}$ . Then  $(\lambda_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$  if and only if  $(\lambda_i)_{i \in \mathbb{N}}$  converges to  $\lambda$  in subsurface projection.*

*Proof.* Let  $W$  be a domain with (possibly empty) geodesic boundary such that  $\lambda \in \mathcal{EL}(W)$ . For every  $i \in \mathbb{N}$  let  $\mu_i \in \pi_W(\lambda_i)$ . Every  $\mu_i$  is either an ending lamination or a collection of curves on  $W$ . We divide  $(\mu_i)_{i \in \mathbb{N}}$  into two subsequences accordingly, and we will prove that each of these sequences superconverges to  $\lambda$  if and only if the corresponding subsequence of  $(\lambda_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ .

Suppose that  $\mu_i \in \mathcal{EL}(W)$  for all  $i \in \mathbb{N}$ . Then  $\lambda_i \cap W = \mu_i$  and it follows immediately that  $(\lambda_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$  if and only if  $(\mu_i)_{i \in \mathbb{N}}$  does.

Now assume that  $\mu_i \subset \mathcal{S}(W)$  for all  $i \in \mathbb{N}$  and  $W \neq \Sigma_{0,5}$ . Let  $a_i \subset \lambda_i \cap W$  be a geodesic arc or curve disjoint from  $\mu_i$ . By Lemma 5.1.6.(ii),  $\mu_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$  if and only if  $a_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$ . If  $(a_i)$  superconverges to  $\lambda$  then  $(\lambda_i)$  superconverges to  $\lambda$  as well. Conversely, if  $(\lambda_i)$  superconverges to  $\lambda$  then  $(\lambda_i \cap W)$  superconverges to  $\lambda$ . For any  $i \in \mathbb{N}$  the set  $\lambda_i \cap W$  is a union of geodesic curves and arcs that belong to finitely many isotopy classes. We conclude from Lemma 5.1.5 and Lemma 5.1.6 that  $(a_i)$  superconverges to  $\lambda$ .

Lastly, consider the case that  $\mu_i \subset \mathcal{S}(W)$  for all  $i \in \mathbb{N}$  and  $W = \Sigma_{0,5}$ . Then  $\mu_i$  consists of a single curve disjoint from  $\lambda_i$ , and Lemma 5.1.6.(i) gives that  $(\mu_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$  if and only if  $\lambda_i^{\text{PT}} \xrightarrow{s} \lambda^{\text{PT}}$ .  $\square$

**Definition 5.1.8.** The *topology of subsurface projection*  $\mathbf{T}_p$  on  $\mathcal{BL}$  is the topological space with the following closed sets: a set  $C \subset \mathcal{BL}$  is closed if and only if for every sequence  $(\lambda_i)_{i \in \mathbb{N}}$  of points in  $C$  that converges in subsurface projection to  $\lambda \in \mathcal{BL}$  we have  $\lambda \in C$ .

In other words,  $\mathbf{T}_p$  is the ‘Fréchet topology’ (see Appendix A) induced by the limit operator defined by convergence in subsurface projection. Recall that  $\mathbf{T}_s$  denotes the topology of superconvergence, defined in Definition 3.2.4.

**Proposition 5.1.9.** *The topologies  $\mathbf{T}_p$  and  $\mathbf{T}_s$  on  $\mathcal{BL}$  are the same.*

*Proof.* Remark 3.2.5 states that the topology  $\mathbf{T}_s$  on  $\mathcal{BL}$  is the Fréchet topology induced by the restriction to  $\mathcal{BL}$  of the limit operator on  $\mathcal{L}(\Sigma_{0,5})$  defined by superconvergence. This limit operator is the same as the limit operator defined by subsurface projection, as a consequence of Proposition 5.1.7.  $\square$

## 5.2 A neighbourhood base for $(\mathcal{BL}, \mathbf{T}_s)$

We will use Proposition 5.1.7 to describe an explicit neighbourhood base for the topology of superconvergence on  $\mathcal{BL}$  at a non-filling lamination. We will refer to this neighbourhood base in the upcoming chapters.

**Lemma 5.2.1.** *Let  $\gamma \in \mathcal{S}(\Sigma_{0,5})$ . If  $\lambda \in \mathcal{BL}$  with  $\iota(\lambda, \gamma) \neq 0$  and  $(\lambda_i)_{i \in \mathbb{N}}$  is a sequence of boundary laminations superconverging to  $\lambda$ , then there exists some  $K \in \mathbb{N}$  such that for all  $i > K$ ,*

$$\iota(\lambda_i, \gamma) \neq 0 \text{ and } \pi_\gamma(\lambda) \subset \pi_\gamma(\lambda_i).$$

*Proof.* Let  $\lambda$  be a boundary lamination that intersects  $\gamma$ . Note that  $\lambda$  intersects  $W_\gamma$  in arcs which are part of finitely many (in fact, at most 3) isotopy classes. Let  $\mathbf{a}$  be an arc in  $\lambda \cap W_\gamma$  and choose  $x^{\text{PT}} \in \mathbf{a}^{\text{PT}}$ . By superconvergence there exist  $x_i \in \lambda_i$  such that  $x_i^{\text{PT}} \rightarrow x^{\text{PT}}$ . Eventually, the geodesic on  $W_\gamma$  extending  $x_i^{\text{PT}}$  is a geodesic arc isotopic to  $\mathbf{a}$ . In particular,  $\iota(\lambda_i, \gamma) > 0$ . Applying this argument to every isotopy class of arcs in  $\lambda \cap W_\gamma$ , we deduce that there exists  $K \in \mathbb{N}$  such that  $\iota(\lambda_i, \gamma) \neq 0$  and  $\pi_\gamma(\lambda) \subset \pi_\gamma(\lambda_i)$  for all  $i > K$ .  $\square$

Let  $\Theta_W$  denote Klarreich's homeomorphism (Theorem 3.2.24.(i)). For notational convenience, we introduce the following map of sets.

**Definition 5.2.2** (Boundary projection). Let  $W$  be a domain in  $\Sigma_{0,5}$  (note that  $W \cong \Sigma_{0,3}^1$  or  $W \cong \Sigma_{0,5}$ ). The *boundary projection to  $W$*  is the map

$$\Pi_W: \mathcal{BL} \rightarrow \mathbb{P}(\overline{\mathcal{C}(W)}), \quad \Pi_W(\lambda) = \begin{cases} \{\Theta_W^{-1}(\lambda)\} \subset \partial \mathcal{C}(W) & \text{if } \text{Supp}(\lambda) = W, \\ \pi_W(\lambda) \subset \mathcal{C}(W) & \text{otherwise.} \end{cases}$$

In the case that  $W = W_\gamma$  for some curve  $\gamma \in \mathcal{S}(\Sigma_{0,5})$  we also write  $\Pi_\gamma = \Pi_{W_\gamma}$ .

Write  $\mathcal{F}_\gamma$  for the Farey graph that is the curve graph of  $W_\gamma$ . Suppose that  $\lambda$  is a boundary lamination supported on  $W_\gamma$ . For a neighbourhood  $U \subset \overline{\mathcal{F}_\gamma}$  of  $\Pi_\gamma(\lambda)$ , we define

$$N(U, \lambda) = \{\mu \in \mathcal{BL} : \Pi_\gamma(\mu) \cap U \neq \emptyset\}.$$

**Lemma 5.2.3.** *Suppose that  $\lambda \in \mathcal{EL}(W_\gamma)$ . If  $U \subset \overline{\mathcal{F}_\gamma}$  is an open neighbourhood of  $\Pi_\gamma(\lambda)$ , then  $N(U, \lambda)$  is an open neighbourhood of  $\lambda$  in  $(\mathcal{BL}, \mathbf{T}_s)$ .*

*Proof.* Suppose that  $U \subset \overline{\mathcal{F}_\gamma}$  is an open neighbourhood of  $\Pi_\gamma(\lambda)$ . It is immediate from the definition that  $\lambda \in N = N(U, \lambda)$  so it suffices to prove that  $N$  is open. We use the fact that  $(\mathcal{BL}, \mathbf{T}_s)$  is a sequential space (Remark 3.2.5 and Lemma A.7.(i)). Suppose that  $(\mu_i)_{i \in \mathbb{N}}$  is a sequence of boundary laminations that superconverges to  $\mu \in N$ . We will show that  $(\mu_i)_{i \in \mathbb{N}}$  is eventually contained in  $N$ .

If  $\text{Supp}(\mu) = W_\gamma$  then for any  $\lambda_i \in \Pi_\gamma(\mu_i)$  the sequence  $(\lambda_i)_{i \in \mathbb{N}}$  superconverges to  $\mu$  by Proposition 5.1.7. Theorem 3.2.24 gives that  $\Pi_\gamma(\mu_i) \rightarrow \Pi_\gamma(\mu)$  in  $\overline{\mathcal{F}_\gamma}$ . Since  $U$  is open,  $\Pi_\gamma(\mu_i)$  is eventually contained in  $U$ , hence  $\mu_i$  is eventually contained in  $N$ .

Now consider the case  $\text{Supp}(\mu) \neq W_\gamma$ , or equivalently  $\iota(\mu, \gamma) \neq 0$ . By Lemma 5.2.1,  $\Pi_\gamma(\mu_i) \supset \Pi_\gamma(\mu)$  hence  $\Pi_\gamma(\mu_i) \cap U \neq \emptyset$  for all sufficiently large  $i$ . It follows that  $\mu_i$  is eventually contained in  $N$ .

We conclude that every sequence that converges to a point in  $N$  is eventually contained in  $N$ . In other words,  $N$  is sequentially open, hence open.  $\square$

**Proposition 5.2.4.** *Suppose that  $\lambda \in \mathcal{EL}(W_\gamma) \subset \mathcal{BL}$  and define  $x = \Theta_{W_\gamma}^{-1}(\lambda) \in \partial \mathcal{F}_\gamma$ . Let  $\mathcal{A}_x$  be the open neighbourhood base at  $x$  from Proposition 2.2.8. The collection*

$$\mathcal{N}_\lambda = \{N(A, \lambda) : A \in \mathcal{A}_x\}$$

*is an open neighbourhood base at  $\lambda$  for the topology of superconvergence on  $\mathcal{BL}$ .*

*Proof.* Let  $\lambda$  be a boundary lamination supported on  $W_\gamma$  and set  $x = \Theta_{W_\gamma}^{-1}(\lambda)$ . By Proposition 2.2.8 and Lemma 5.2.3,  $\mathcal{N}_\lambda$  consists of open neighbourhoods of  $\lambda$ .

We claim that for any open set  $O \subset (\mathcal{BL}, \mathbf{T}_s)$  containing  $\lambda$  there is some  $N \in \mathcal{N}_\lambda$  such that  $N \subset O$ . We prove this claim by assuming the contrary. Suppose that  $O \subset (\mathcal{BL}, \mathbf{T}_s)$  is an open set containing  $\lambda$  that does not contain any element of  $\mathcal{N}_\lambda$ . Take a subcollection  $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{A}_x$  with the property that  $A_i \supset A_{i+1}$  for all  $i \in \mathbb{N}$ . For every  $i \in \mathbb{N}$  there is some  $\mu_i \in N(A_i, \lambda)$  such that  $\mu_i \notin O$ . Then  $\Pi_\gamma(\mu_i) \cap A_j \neq \emptyset$  for all  $i \geq j$ , hence  $(\Pi_\gamma(\mu_i))_{i \in \mathbb{N}}$  converges to  $\Pi_\gamma(\lambda)$  (Proposition 2.2.8). By Proposition 5.1.7 the sequence  $(\mu_i)_{i \in \mathbb{N}}$  superconverges to  $\lambda$ . This contradicts the assumption that  $O$  is open, proving the claim. It follows that  $\mathcal{N}_\lambda$  is a neighbourhood base at  $\lambda$ .  $\square$

### 5.3 Relating boundary laminations to $\partial\mathcal{P}(\Sigma_{0,5})$

The goal of this section is to deduce Corollary 5.3.5. This states that  $\partial\mathcal{P}(\Sigma_{0,5})$  and  $(\mathcal{BL}, \mathbf{T}_s)$  are homeomorphic.

Let  $\mathrm{WP}(\Sigma)$  denote the Teichmüller space of  $\Sigma_{0,5}$  with the Weil-Petersson metric and write  $\mathrm{WP}^*(\Sigma)$  for its completion, known as the ‘augmented Teichmüller space’. For an introduction to the topic, see [Wol1, Wol2]. It is well known that  $\mathrm{WP}(\Sigma)$  has the structure of a  $\mathrm{CAT}(0)$ -space and  $\mathrm{WP}^*(\Sigma)$  is not locally compact. We recall a couple of theorems regarding the Teichmüller space, due to Brock and Farb [BroCF, Theorems 1.1 and 1.5]. The second theorem was proven by Brock for closed surfaces [BroCF, Theorem 1.1], and is commonly attributed to Brock also in the punctured case.

**Theorem 5.3.1** (Brock-Farb).  *$\mathrm{WP}(\Sigma)$  is  $\delta$ -hyperbolic if and only if  $\xi(\Sigma) \leq 2$ .*

**Theorem 5.3.2** (Brock).  *$\mathcal{P}(\Sigma)$  and  $\mathrm{WP}(\Sigma)$  are quasi-isometric.*

To identify the Gromov and  $\mathrm{CAT}(0)$ -boundaries of the augmented Teichmüller space we need the following general fact.

**Theorem 5.3.3** (Buckley-Falk). *Let  $X$  be a  $\delta$ -hyperbolic and complete  $\mathrm{CAT}(0)$ -space. There is a homeomorphism between the Gromov and  $\mathrm{CAT}(0)$ -boundaries of  $X$ .*

A proof of this result under the extra assumption that  $X$  is proper can be found in Bridson and Haefliger’s book [BriH, Proposition 3.7.2]. However, since  $\mathrm{WP}^*(\Sigma)$  is not locally compact it is not proper either, so we cannot appeal to this proof. In the general case the result follows from [BuF, Theorems 1.1 and 1.2].

Brock and Masur proved the following theorem [BroCM, Theorem 5].

**Theorem 5.3.4** (Brock-Masur).  *$(\mathcal{BL}, \mathbf{T}_p)$  is homeomorphic to the  $\mathrm{CAT}(0)$ -boundary of  $\mathrm{WP}^*(\Sigma_{0,5})$ .*

Using these theorems and Proposition 5.1.9 we draw the following conclusion.

**Corollary 5.3.5.** *There is a homeomorphism  $\partial\mathcal{P}(\Sigma_{0,5}) \rightarrow (\mathcal{BL}, \mathbf{T}_s)$ .*

*Proof.* Write  $\Sigma = \Sigma_{0,5}$ . By Theorem 5.3.2,  $\mathcal{P}(\Sigma)$  is quasi-isometric to  $\mathrm{WP}(\Sigma)$ . The inclusion  $\mathrm{WP}(\Sigma) \subset \mathrm{WP}^*(\Sigma)$  is a quasi-isometry. Theorem 5.3.1 gives that  $\mathrm{WP}(\Sigma)$  is hyperbolic. It follows that  $\mathcal{P}(\Sigma)$  and  $\mathrm{WP}^*(\Sigma)$  are hyperbolic as well, and the quasi-isometry induces a homeomorphism between Gromov boundaries  $\partial\mathcal{P}(\Sigma) \cong \partial\mathrm{WP}^*(\Sigma)$  [BriH, Theorems III.H.1.9 and 3.9]. Theorem 5.3.3 says that there is a homeomorphism from  $\partial\mathrm{WP}^*(\Sigma)$  to the  $\mathrm{CAT}(0)$ -boundary of  $\mathrm{WP}^*(\Sigma)$ . The latter is homeomorphic to  $(\mathcal{BL}, \mathbf{T}_p)$  by Theorem 5.3.4. The result now follows from Proposition 5.1.9.  $\square$

## 5.4 Convergence in measure

In the final section of this chapter we prove Proposition 5.4.4. Informally, this says that the topology of superconvergence on  $\mathcal{BL}$  is not induced by the measure topology.

Recall that  $\Psi: \mathcal{ML}(\Sigma_{0,5}) - \{(\emptyset, 0)\} \rightarrow \mathcal{PML}(\Sigma_{0,5})$  is the projectivisation map (page 29) and  $\Phi: \mathcal{PML}(\Sigma_{0,5}) \rightarrow \mathcal{UML}(\Sigma_{0,5})$  is the measure-forgetting map (page 29). Furthermore,  $(\mathcal{P})(\mathcal{A})\mathcal{BL}$  denotes the set of (projective) (almost) boundary laminations (Definitions 3.2.11, 3.2.12 and 3.2.25). Recall that if  $\lambda$  is an almost boundary lamination, then  $\lambda^*$  denotes the sublamination of  $\lambda$  that is a boundary lamination.

The following statement is due to Gabai [Gab1, Proposition 3.2].

**Proposition 5.4.1** (Gabai). *Suppose that  $(x_i)_{i \in \mathbb{N}}$  is a sequence of measured laminations on  $\Sigma_{0,5}$  converging to  $x \in \mathcal{ML}(\Sigma_{0,5})$ . If  $\Phi\Psi(x) \in \mathcal{ABL}$  and  $\Phi\Psi(x_i) \in \mathcal{ABL}$  for all  $i \in \mathbb{N}$ , then  $(\Phi\Psi(x_i)^*)_{i \in \mathbb{N}}$  superconverges to  $\Phi\Psi(x)^*$ .*

Propositions 3.2.18 and 5.4.1 can also be stated in terms of projective laminations.

**Corollary 5.4.2.** *Let  $\Sigma$  be a surface.*

- (i) *Suppose that the sequence  $(x_i)_{i \in \mathbb{N}}$  of projective laminations on  $\Sigma$  converges to  $x \in \mathcal{PML}(\Sigma)$ . Then  $(\Phi(x_i))_{i \in \mathbb{N}}$  superconverges to  $\Phi(x)$ .*
- (ii) *If in addition  $\Sigma = \Sigma_{0,5}$ ,  $\Phi(x) \in \mathcal{ABL}$  and  $\Phi(x_i) \in \mathcal{ABL}$  for all  $i \in \mathbb{N}$ , then  $(\Phi(x_i)^*)_{i \in \mathbb{N}}$  superconverges to  $\Phi(x)^*$ .*

*Proof.* This follows immediately from Propositions 3.2.18 and 5.4.1 and the existence of a section of  $\Psi$  (Remark 3.3.19).  $\square$

**Remark 5.4.3.**  $\Phi^{-1}(\mathcal{EL}(\Sigma))$  is dense in  $\mathcal{PML}(\Sigma)$ . Even more, every  $\text{MCG}(\Sigma)$ -orbit of projective laminations is dense in  $\mathcal{PML}(\Sigma)$  [FatLP, Theorem 6.19].

We use Corollary 5.4.2 and Remark 5.4.3 to deduce the following.

**Proposition 5.4.4.**

- (i) *The assignment  $\Omega(x) = \Phi(x)^*$  defines a continuous map*

$$\Omega: \mathcal{PABL} \rightarrow (\mathcal{BL}, \mathbf{T}_s).$$

- (ii) *The maps  $\Omega$  and  $\Omega|_{\mathcal{PBL}}$  are not quotient maps.*

*Proof.* We start with part (i). Since  $\mathcal{PML}(\Sigma_{0,5})$  is metrisable,  $\Omega$  is continuous if and only if it is sequentially continuous. But this is clear from Corollary 5.4.2.(ii).

To prove part (ii), it suffices to find a set  $A \subset \mathcal{PBL}$  that is closed as a subset of  $\mathcal{PABL}$ , with the property that  $\Omega(A)$  is not closed and  $A = \Omega^{-1}\Omega(A)$ . For this we use the following claim.

**Claim.** *Let  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  and  $\lambda \in \mathcal{EL}(W_\alpha)$ . There exist  $x_i \in \mathcal{PML}(\Sigma_{0,5})$  with  $\Phi(x_i)$  uniquely ergodic and ending for every  $i \in \mathbb{N}$  such that  $x_i \rightarrow \alpha$  in  $\mathcal{PML}(\Sigma_{0,5})$  and  $\Phi(x_i) \xrightarrow{s} \lambda$ .*

*Proof of claim.* By Remark 3.3.21  $\lambda$  has a unique preimage in  $\mathcal{PML}(\Sigma_{0,5})$ , which we will also denote by  $\lambda$ . Consider the set

$$\{t\lambda + (1-t)\alpha : t \in \mathbb{I}\} \subset \mathcal{PML}(\Sigma_{0,5}).$$

For every  $t \in \mathbb{I}$  there exists a sequence  $(x_i)_{i \in \mathbb{N}}$  in  $\mathcal{PML}(\Sigma_{0,5})$  which converges to  $t\lambda + (1-t)\alpha$ , such that the underlying lamination of every  $x_i$  is uniquely ergodic and ending (Remark 5.4.3). Corollary 5.4.2 gives that  $\Phi(x_i) \xrightarrow{s} \lambda$ . For every  $n \in \mathbb{N}$  take a sequence  $(x_i^n)_{i \in \mathbb{N}}$  converging to  $\frac{1}{n}\lambda + (1 - \frac{1}{n})\alpha$ .

The proof is completed by taking a diagonal sequence. Let  $\{U_k\}_{k \in \mathbb{N}}$  be a countable neighbourhood base at  $\alpha$  for the topology of  $\mathcal{PML}(\Sigma_{0,5})$  with the property that  $U_{k+1} \subset U_k$  for all  $k \in \mathbb{N}$ . Define  $i(0) = 0$ . For every  $k \in \mathbb{N}$  take  $i(k), n(k) \in \mathbb{N}$  such that  $i(k) > i(k-1)$  and  $y_k = x_{i(k)}^{n(k)} \in U_k$ . Then the sequence  $(y_k)_{k \in \mathbb{N}}$  converges to  $\alpha$  and  $(\Phi(y_k))_{k \in \mathbb{N}}$  superconverges to  $\lambda$ .  $\square$

Let  $(x_i)_{i \in \mathbb{N}}$  and  $\lambda$  be as in the claim. Define  $A = \{x_i : i \in \mathbb{N}\}$ . Note that  $A$  is a subset of  $\mathcal{PBL}$  that is closed as a subset of  $\mathcal{PABL}$ . Since every  $x_i$  is uniquely ergodic,  $\Omega^{-1}\Omega(A) = A$ . On the other hand,  $\Omega(A)$  accumulates on  $\lambda \notin \Omega(A)$ , so  $\Omega(A)$  is not closed.  $\square$



## Chapter 6

# Connectivity properties of $(\mathcal{BL}, \mathbf{T}_s)$

In this chapter we further investigate the topology of superconvergence on the set of boundary laminations.

In Section 6.1 we observe that  $(\mathcal{BL}, \mathbf{T}_s)$  is connected (Proposition 6.1.2). Using different methods, we show in Section 6.2 that  $(\mathcal{BL}, \mathbf{T}_s)$  is path connected (Proposition 6.2.3).

The last three sections of this chapter are dedicated to proving that  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected (Theorem 6.5.3). Explicitly, Section 6.3 finds particular train tracks (Lemma 6.3.1) with good carrying properties (Theorem 6.3.3). In Section 6.4 we show that any two points in  $(\mathcal{BL}, \mathbf{T}_s)$  can be connected by a path which projects near a geodesic in  $\mathcal{F}_\gamma$  (Theorem 6.4.3). Finally, in Section 6.5 we prove the main result of this chapter, Theorem 6.5.3. The proof relies on the fact that  $(\mathcal{EL}, \mathbf{T}_s)$  is locally path connected (Theorem 3.2.10).

### 6.1 Connectivity of $(\mathcal{BL}, \mathbf{T}_s)$

Consider the five-times punctured sphere, endowed with a fixed hyperbolic structure. Throughout the chapter we will leave  $\Sigma_{0,5}$  implicit and write  $\mathcal{S}, \mathcal{C}, \mathcal{P}, \mathcal{PML}, \mathcal{EL}$  to denote the respective objects associated to  $\Sigma_{0,5}$ . See Chapter 3 for the relevant definitions.

Adopting the notation from the previous chapters, let  $W_\gamma$  denote the domain in  $\Sigma_{0,5}$  whose boundary curve is  $\gamma \in \mathcal{S}$  (page 26). The curve graph of  $W_\gamma$  is a Farey graph  $\mathcal{F}_\gamma$  (page 79).

As a warm-up for further analysis of  $(\mathcal{BL}, \mathbf{T}_s)$ , we show that it is connected.

This is a consequence of the following elementary observation.

**Lemma 6.1.1.**  $\mathcal{EL}$  is a dense subset of  $(\mathcal{BL}, \mathbf{T}_s)$ .

*Proof.*  $\Phi^{-1}(\mathcal{EL})$  is dense as a subset of  $\mathcal{PML}$  (Remark 5.4.3), and therefore also as a subset of  $\mathcal{PABL}$ . Since a continuous surjection maps dense sets to dense sets, Proposition 5.4.4.(i) implies that  $\mathcal{EL}$  is a dense subset of  $(\mathcal{BL}, \mathbf{T}_s)$ .  $\square$

Combining Lemma 6.1.1 with the fact that  $(\mathcal{EL}, \mathbf{T}_s)$  is connected (Theorem 3.2.10) we obtain the following.

**Proposition 6.1.2.**  $(\mathcal{BL}, \mathbf{T}_s)$  is connected.

*Proof.* The subset  $\mathcal{EL} \subset (\mathcal{BL}, \mathbf{T}_s)$  is dense (Lemma 6.1.1) and the subspace topology on  $\mathcal{EL}$  is the topology of superconvergence. Theorem 3.2.10 shows that  $(\mathcal{EL}, \mathbf{T}_s)$  is connected. Any space that has a connected dense subset is connected itself. This proves the proposition.  $\square$

## 6.2 Path connectivity of $(\mathcal{BL}, \mathbf{T}_s)$

In this section we improve the result of Proposition 6.1.2 and show that  $(\mathcal{BL}, \mathbf{T}_s)$  is path connected. Moreover, the proof presented in this section does not rely on the connectivity of  $(\mathcal{EL}, \mathbf{T}_s)$ .

By Proposition 5.4.4.(i) the map  $\Omega: \mathcal{PABL} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  that sends a projective almost boundary lamination to the boundary lamination it contains is continuous. Therefore, to prove that  $(\mathcal{BL}, \mathbf{T}_s)$  is path connected it suffices to show that  $\mathcal{PABL}$  is path connected.

We will use the notation set up in Section 4.1. Recall that  $\eta$  is the inclusion of  $\mathcal{C}$  into  $\mathcal{PML}$  (Definition 4.1.1),  $D_\alpha \subset \mathcal{PML}$  is the PL disc consisting of all projective laminations disjoint from  $\alpha \in \mathcal{S}$  and  $D_{\alpha,\beta} = D_\alpha \cap D_\beta$  when  $\alpha \neq \beta$  (Definition 4.1.3 and Remark 4.1.7). It follows that

$$\eta(\mathcal{C}) = \bigcup_{\alpha, \beta \in \mathcal{S}} D_{\alpha, \beta} = \mathcal{PML} - \mathcal{PABL}.$$

Recall the following definition.

**Definition 6.2.1** (Local path connectivity). Let  $X$  be a topological space.

- Suppose that  $x \in X$ . Say that  $X$  is *locally path connected at  $x$*  if for every neighbourhood  $N \subset X$  of  $x$  there exists a neighbourhood  $M \subset X$  of  $x$  such that any  $y \in M$  can be connected to  $x$  by a path  $p: I \rightarrow X$  with image in  $N$ .

- $X$  is *locally path connected* if  $X$  is locally path connected at every  $x \in X$ .

The following lemma relies on general facts about complements of countably many intervals in  $\mathbb{R}^3$ . We have included a discussion of this in Appendix B.

**Lemma 6.2.2.**

- (i) *If  $\tau$  is a complete train track on  $\Sigma_{0,5}$ , then  $P(\tau) \cap \mathcal{PABL}$  is path connected and locally path connected.*
- (ii) *Suppose that  $\lambda, \mu \in \mathcal{PABL}$  and  $p: I \rightarrow \mathcal{PML}$  is a path from  $\lambda$  to  $\mu$ . For any neighbourhood  $N \subset \mathcal{PML}$  of  $p$  there exists a path in  $N \cap \mathcal{PABL}$  from  $\lambda$  to  $\mu$ .*
- (iii)  *$\mathcal{PABL}$  is path connected and locally path connected.*

*Proof.* We start with part (i). Let  $\tau$  be a complete train track on the five-times punctured sphere. For every pair of disjoint curves  $\alpha, \beta \in \mathcal{S}$  the set  $D_{\alpha, \beta}$  intersects  $P(\tau)$  in one of the following ways. If neither  $\alpha$  nor  $\beta$  is carried by  $\tau$ , then  $D_{\alpha, \beta} \cap P(\tau)$  is empty. If both  $\alpha$  and  $\beta$  are carried by  $\tau$ , then  $D_{\alpha, \beta}$  is an interval contained in  $P(\tau)$ . Otherwise,  $D_{\alpha, \beta} \cap P(\tau)$  is a single point in the frontier of  $P(\tau)$ . We conclude that  $\eta(\mathcal{C})$  intersects  $P(\tau)$  in a countable union of intervals and points. It follows from Corollary B.4 that  $P(\tau) - \eta(\mathcal{C})$  is path connected and locally path connected, proving part (i).

We proceed with part (ii). Let  $\mathcal{K}$  be a CW structure on  $\mathcal{PML}$  whose 3-cells are the train track polytopes of a collection of maximal standard tracks  $\{\tau_1, \tau_2, \dots, \tau_n\}$ , as in Corollary 3.3.18. Suppose that  $\lambda, \mu \in \mathcal{PABL}$  are connected by a path  $p: I \rightarrow \mathcal{PML}$  and  $N$  is a neighbourhood of  $p$ . Up to replacing  $p$  by a nearby path, we may assume that  $p$  avoids the 1-skeleton of  $\mathcal{K}$  and intersects the 2-skeleton of  $\mathcal{K}$  in finitely many points. Split  $I$  into finitely many segments  $[t_i, t_{i+1}]$  such that  $p(t) \prec \tau_{k_i}$  for all  $t \in [t_i, t_{i+1}]$  and  $k_i \neq k_{i+1}$  for all  $i$ . Observe that  $p(t_i) \in P(\tau_{k_{i-1}}) \cap P(\tau_{k_i})$ . If  $p(t_i) \in \mathcal{PABL}$ , define  $q(t_i) = p(t_i)$ ; if not, let  $q(t_i)$  be a projective almost boundary lamination that lies in the component of  $N \cap P(\tau_{k_{i-1}}) \cap P(\tau_{k_i})$  containing  $p(t_i)$ . Such a lamination exists, since the complement of countably many intervals and lines is dense in dimension 2 (Lemma B.1). By part (i), we can connect  $q(t_i)$  and  $q(t_{i+1})$  by a path in  $N \cap P(\tau_{k_i})$ . Concatenate these paths to find the required path  $q$  connecting  $\lambda$  and  $\mu$  within  $N \cap \mathcal{PABL}$ .

It remains to prove part (iii). Path connectivity of  $\mathcal{PABL}$  is an immediate consequence of (ii). To show that  $\mathcal{PABL}$  is also locally path connected, suppose that  $\lambda \in \mathcal{PABL}$  and take a neighbourhood  $N \subset \mathcal{PML}$  of  $\lambda$ . Let  $M$  be a path connected neighbourhood of  $\lambda$  in  $\mathcal{PML}$  such that  $\text{cl } M \subset \text{int } N$ . By part (ii), every

$\mu \in M \cap \mathcal{PABL}$  can be connected to  $\lambda$  by a path in  $N \cap \mathcal{PABL}$ . This completes the proof of part (iii) and the proof of the lemma.  $\square$

We use Lemma 6.2.2 to deduce the following. Note that the proof does not rely on the path connectivity of  $\mathcal{EL}$ .

**Proposition 6.2.3.**  *$(\mathcal{BL}, \mathbf{T}_s)$  is path connected.*

*Proof.*  $\mathcal{PABL}$  is path connected by Lemma 6.2.2.(iii). Applying Proposition 5.4.4.(i) we find that  $\Omega(\mathcal{PABL}) = (\mathcal{BL}, \mathbf{T}_s)$  is path connected as well.  $\square$

We also prove something slightly stronger.

**Proposition 6.2.4.** *For every  $\gamma \in \mathcal{S}$  the complement of  $\mathcal{EL}(W_\gamma)$  in  $(\mathcal{BL}, \mathbf{T}_s)$  is path connected.*

*Proof.* Since  $D_\gamma$  is a PL disc in  $\mathcal{PML}$  (Remark 4.1.7), its complement is open and path connected. It follows from Lemma 6.2.2.(ii) that between any  $\lambda, \mu \in \mathcal{PABL} - D_\gamma$  there is a path in  $\mathcal{PABL}$  that avoids  $D_\gamma$ . Hence  $\mathcal{PABL} - D_\gamma$  is path connected, and therefore  $\Omega(\mathcal{PABL} - D_\gamma) = (\mathcal{BL} - \mathcal{EL}(W_\gamma), \mathbf{T}_s)$  is.  $\square$

### 6.3 A track carrying all boundary laminations

Let  $\gamma$  be a curve on  $\Sigma_{0,5}$ . We will describe one train track  $\tau$  that carries every boundary lamination, up to the action of the mapping class group. More specifically, associated to  $\tau$  there is a curve  $\gamma \in \mathcal{S}$  such that any boundary lamination that intersects  $\gamma$  is carried by  $f\tau$ , for some  $f \in \text{MCG}(\Sigma_{0,5})$  with  $f(\gamma) = \gamma$ .

As in Example 3.1.1, let  $T_\gamma$  and  $H_\gamma$  denote the Dehn twist and half Dehn twist about a curve  $\gamma$ , respectively. We use the notation  $\text{MCG}(\Sigma_{0,5})_\gamma < \text{MCG}(\Sigma_{0,5})$  to denote the stabiliser of  $\gamma$  with respect to the action of  $\text{MCG}(\Sigma_{0,5})$  on  $\mathcal{C}$ . Observe that  $\text{MCG}(W_\gamma) < \text{MCG}(\Sigma_{0,5})_\gamma$  is a subgroup of index 2 and the quotient is generated by  $H_\gamma$ .

**Lemma 6.3.1.** *Let  $\tau$  be a train track on  $\Sigma_{0,5}$  consisting of five branches  $b_1, \dots, b_5$  and one switch  $s$ , with the following combinatorics at  $s$ . If  $h_i^-, h_i^+$  are the half-branches making up  $b_i$ , then the partition of the half-branches at  $s$  is*

$$\{\{h_1^\pm, h_2^\pm, h_3^\pm\}, \{h_4^\pm, h_5^\pm\}\}.$$

*Furthermore, the order in which a small loop surrounding  $s$  passes through the incident half-branches is*

$$h_1^-, h_2^-, h_2^+, h_3^-, h_3^+, h_1^+, h_4^-, h_4^+, h_5^-, h_5^+.$$

The following is true.

- (i)  $\tau$  exists and is unique up to the action of  $\text{MCG}(\Sigma_{0,5})$ .
- (ii) There exists a curve  $\gamma \in \mathcal{S}$  such that every embedded curve that hits  $\tau$  efficiently and intersects  $\tau$  once is a representative of  $\gamma$ .
- (iii)  $\tau$  is complete.

*Proof.* To show that  $\tau$  exists and is unique, note that the combinatorics at  $s$  completely determine a pretrack on the sphere. The complement of this pretrack consist of five monogons and one trigon. Therefore, there is exactly one way to place five punctures so that the pretrack becomes a train track on  $\Sigma_{0,5}$ .

Suppose that  $g$  is an embedded curve that intersects  $\tau$  efficiently in one point  $x$ . Then  $g - \{x\}$  is an arc in a complementary component of  $\tau$  whose endpoints are transverse to  $\tau$ . It follows that  $x = s$  and  $g - \{s\}$  lies in a once punctured monogon. On a once punctured monogon all arcs that do not bound a bigon are isotopic, from which it follows that the isotopy class of  $g$  is independent on the choice of  $g$ . Let  $\gamma \in \mathcal{C}$  be this isotopy class. This curve is illustrated in Figure 6.1.

To show that  $\tau$  is complete, we need to observe that it is recurrent and transversely recurrent. This is an easy exercise, performed in Figure 6.2.  $\square$

Reversing the roles of  $\gamma$  and  $\tau$  we obtain the following statement.

**Corollary 6.3.2.** *For every  $\gamma \in \mathcal{S}$  there exists a unique  $\text{MCG}(\Sigma_{0,5})_\gamma$ -orbit of train tracks, denoted  $T(\gamma)$ , such that every  $\tau \in T(\gamma)$  and  $\gamma$  satisfy Lemma 6.3.1.*

*Proof.* Fix  $\gamma \in \mathcal{S}$ . Take any train track  $\tau$  satisfying Lemma 6.3.1. Up to applying a mapping class, we may assume that every embedded curve that hits  $\tau$  efficiently and intersects  $\tau$  once represents  $\gamma$ . By Lemma 6.3.1.(i) any train track  $\sigma$  that has the same properties there is a mapping class  $f$  with  $f\sigma = \tau$ . Observe that  $f\gamma = \gamma$ , hence  $f \in \text{MCG}(\Sigma_{0,5})_\gamma$ .  $\square$

For a triangle  $\Delta$  in  $\mathcal{F}_\gamma$ , let

$$R_\Delta \in \text{MCG}(W_\gamma)$$

be a mapping class that permutes the punctures of  $W_\gamma$  in such a way that  $R_\Delta^3 = T_\gamma$  and  $R_\Delta(\Delta) = \Delta$ . Note that  $R_\Delta$  acts on  $\mathcal{F}_\gamma$  by rotating about  $\Delta$ .

Recall that  $\pi_\gamma = \pi_{W_\gamma}$  denotes the subsurface projection to  $W_\gamma$  (Remark 3.4.8). The subsurface projection of a train track is defined as the subsurface projection

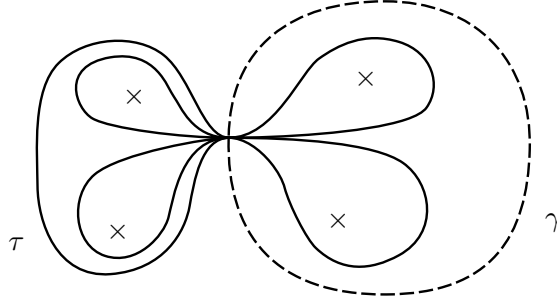


Figure 6.1: The train track  $\tau$  and the curve  $\gamma$  on  $\Sigma_{0,5}$ . One puncture is at infinity.

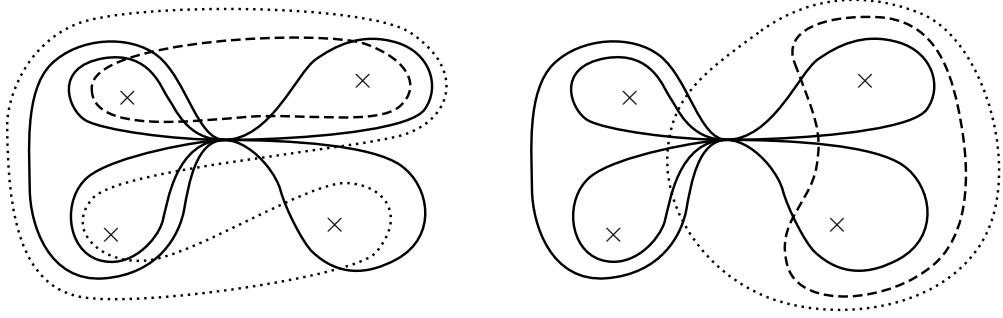


Figure 6.2: The train track  $\tau$  is recurrent (left) and transversely recurrent (right).

of its vertex cycles (Definition 3.4.6). Observe that for any  $\tau \in T(\gamma)$  the set  $\pi_\gamma(\tau)$  is a triangle in  $\mathcal{F}_\gamma$ . If  $\sigma, \tau \in T(\gamma)$  with  $\pi_\gamma(\sigma) = \pi_\gamma(\tau) = \Delta$  then  $\tau = f\sigma$  for some  $f \in \langle R_\Delta, H_\gamma \rangle < \text{MCG}(\Sigma_{0,5})_\gamma$ .

The following theorem shows that every boundary lamination intersecting  $\gamma$  is carried by some  $\tau \in T(\gamma)$ .

**Theorem 6.3.3.** *Let  $\gamma \in \mathcal{S}$  and let  $T(\gamma)$  be the collection of train tracks provided by Corollary 6.3.2. For any  $\lambda \in \mathcal{UL}(\Sigma_{0,5})$  the following is true.*

- (i) *If  $\iota(\lambda, \gamma) = 0$  then  $\lambda$  is not carried by any  $\tau \in T(\gamma)$ .*
- (ii) *If  $\iota(\lambda, \gamma) \neq 0$ ,  $\lambda \in \mathcal{BL}$  and  $\Delta$  is a triangle in  $\mathcal{F}_\gamma$  such that  $\pi_\gamma(\lambda) \subset \Delta$ , then  $\lambda$  is carried by a train track  $\tau \in T(\gamma)$  with  $\pi_\gamma(\tau) = \Delta$ .*

*Proof.* Let  $\gamma$  be a curve on  $\Sigma_{0,5}$ , identified with its geodesic representative. Write  $W = W_\gamma$  and  $V = \text{cl}(\Sigma_{0,5} - W)$ . The punctures of  $W$  are labelled 0, 1, 2 and we write 3, 4 for the punctures of  $V$ .

Start with part (i). Fix  $\tau \in T(\gamma)$  and suppose that  $\lambda$  is a lamination with  $\iota(\lambda, \gamma) = 0$  and  $\lambda \prec \tau$ . Let  $\sigma$  be a birecurrent train track on  $W$  that carries  $\lambda$ . By Theorem 3.3.14, there exists a train track  $\rho$  such that  $\rho \prec \sigma$ ,  $\rho \prec \tau$  and  $\lambda \prec \rho$ . The

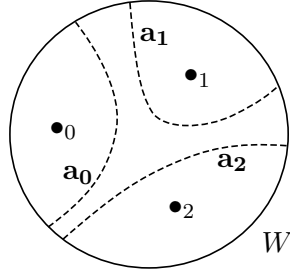


Figure 6.3: A maximal set of pairwise disjoint and non-isotopic arcs on  $W$ .

vertex cycles of  $\rho$  are curves carried by both  $\sigma$  and  $\tau$ . In particular, these curves are disjoint from  $\gamma$ .

We obtain a contradiction by showing that every curve carried by  $\tau$  intersects  $\gamma$ . Suppose that  $\alpha \in \mathcal{S}$  is carried by  $\tau$ . There exists a realisation  $a$  of  $\alpha$  that is contained in a tie neighbourhood of  $\tau$  and is transverse to the ties. Observe that  $a$  does not form any bigons with  $\gamma$  and intersects  $\gamma$  at least twice. It follows from Lemma 3.1.4 that  $\iota(\alpha, \gamma) > 0$ . This completes the proof of part (i).

We proceed with part (ii). Let  $\lambda$  be a boundary lamination that intersects  $\gamma$  non-trivially. Then  $\lambda$  intersects  $W$  and  $V$  in arcs with endpoints on  $\gamma$ . Since  $\lambda$  consists of geodesics and  $W$  and  $V$  have geodesic boundary, all of these arcs are essential. Every essential arc on a thrice-punctured disc divides it into a once-punctured bigon and a twice-punctured bigon – we say the arc is *incident* to these bigons. There are at most three essential pairwise disjoint and non-isotopic arcs on  $W$ . Furthermore, for any two triples  $A, B \in \mathcal{A}(W)$  of disjoint arcs and any  $\alpha \in A$ ,  $\beta \in B$ , there exists  $f \in \text{MCG}(W)$  such that  $f(\alpha) = \beta$  and  $f(A) = B$ . Choose three disjoint arcs  $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$  on  $W$  such that  $\mathbf{a}_i$  is incident to a once-punctured bigon containing the  $i$ th puncture, as depicted in Figure 6.3, and let  $\alpha_i \in \mathcal{S}(W)$  be the curve disjoint from  $\mathbf{a}_i$ . Let  $\Delta \subset \mathcal{F}_\gamma$  be a triangle such that  $\pi_\gamma(\lambda) \subset \Delta$ . Up to applying a mapping class of  $W$ , we may assume that the intersection of  $\lambda$  and  $W$  contains an arc isotopic to  $\mathbf{a}_0$  and  $\Delta = \{\alpha_0, \alpha_1, \alpha_2\}$ .

If  $0 \leq i \leq 2$  such that there exists an arc in  $\lambda \cap W$  isotopic to  $\mathbf{a}_i$ , let  $\mathbf{b}_i$  be the arc in  $\lambda \cap W$  isotopic to  $\mathbf{a}_i$  that is incident to a once-punctured bigon in  $W - \lambda$ . For the remaining  $0 < i \leq 2$ , choose  $\mathbf{b}_i$  to be a geodesic arc isotopic to  $\mathbf{a}_i$  that is disjoint from  $\lambda$  and all previously defined  $\mathbf{b}_j$ . Define  $m_W = \mathbf{b}_0 \cup \mathbf{b}_1 \cup \mathbf{b}_2$  and  $\ell_W = m_W \cap \lambda$ . Let  $P$  be the component of  $\gamma - \lambda$  such that  $P \cup \mathbf{b}_0$  bounds a once-punctured bigon. See Figure 6.4 (left).

All essential arcs on  $V$  are isotopic, so  $\lambda$  intersects  $V$  in arcs of a single isotopy type. There are exactly two arcs in  $\lambda \cap V$  that are incident to a bigon in

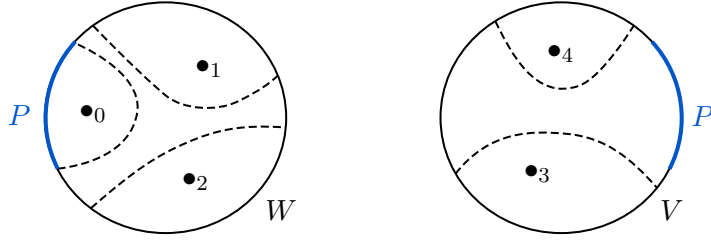


Figure 6.4: The arcs  $m_W \subset W$ ,  $\ell_V \subset V$  and the segment  $P$  of  $\gamma$ .

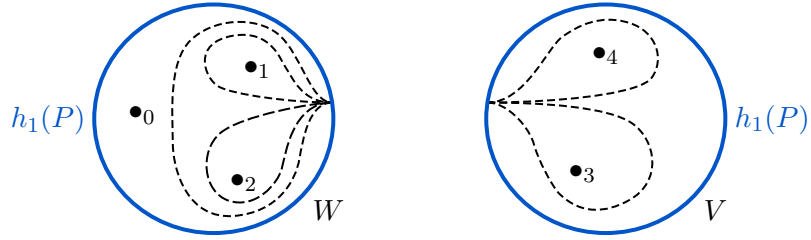


Figure 6.5: There exists a  $C^1$ -map  $h_1$  homotopic to the identity that contracts the complement of  $P$  to a point and maps the arcs of  $m_W$  and  $\ell_V$  to branches of a train track with one switch.

$V - \lambda$ . Let  $\ell_V \subset V$  be the union of these arcs. Observe that  $\ell_V$  cuts  $V$  into two once punctured bigons and a 4-gon  $D$ . We claim that  $P$  is contained in the boundary of  $D$ . Suppose not, then the endpoints of  $P$  are the endpoints of one of the arcs in  $\ell_W$ . They are also the endpoints of the arc  $\mathbf{b}_0$ , hence  $\lambda$  contains a simple closed curve, contradicting the assumption that  $\lambda$  is a boundary lamination. Up to applying a finite number of half Dehn twists if necessary, we may assume that  $P$  intersects  $V$  in the configuration depicted in Figure 6.4 (right).

We use a homotopy that shrinks  $\gamma - P$  to a point to find a train track that carries  $\lambda$ . Let

$$H: \mathbb{I} \times \Sigma_{0,5} \rightarrow \Sigma_{0,5}$$

be a homotopy such that for all  $t \in \mathbb{I}$ ,  $h_t = H(t, *)$  is a  $C^1$ -map that is the identity outside of an annulus with core curve  $\gamma$  and  $h_t(\gamma) = \gamma$ . Furthermore, we require that  $h_0 = \text{Id}$ ,  $h_t$  is a homeomorphism for  $0 \leq t < 1$ ,  $h_1(\gamma - P)$  is a single point and  $h_1$  is a homeomorphism when restricted to  $(\Sigma_{0,5} - \gamma) \cup P$ . This homotopy is illustrated in Figure 6.5. Observe that  $h_1(m_W \cup \ell_V) = \tau$  is a train track that carries  $\lambda$ . Moreover,  $\tau$  satisfies all the properties of Lemma 6.3.1, hence  $\tau \in T(\gamma)$ . Note that  $\pi_\gamma(\tau) = \Delta$ .  $\square$



**Corollary 6.3.4.** *Suppose that  $\gamma \in \mathcal{S}$  and  $\tau \in T(\gamma)$ . If  $\lambda \in \mathcal{BL}$  is disjoint from  $\gamma$  and*

$$\pi_\gamma(\lambda) \subset \pi_\gamma(\tau),$$

*then  $\lambda \prec f\tau$  for some  $f \in \langle R_{\pi_\gamma(\tau)}, H_\gamma \rangle < \text{MCG}(\Sigma_{0,5})_\gamma$ .*

*Proof.* Let  $\gamma, \tau$  and  $\lambda$  be as in the statement and define  $\Delta = \pi_\gamma(\tau)$ . By Theorem 6.3.3.(ii) the lamination  $\lambda$  is carried by a train track  $\sigma \in T(\gamma)$  with  $\pi_\gamma(\sigma) = \Delta$ . Then  $\sigma = f\tau$  for some  $f \in \langle R_\Delta, H_\gamma \rangle < \text{MCG}(\Sigma_{0,5})_\gamma$ , hence  $\lambda \prec f\tau$ .  $\square$

## 6.4 Paths that project near geodesics

The main result of this section is Theorem 6.4.3. This theorem finds paths in  $(\mathcal{BL}, \mathbf{T}_s)$  that project near geodesics in the Farey graph.

If  $\alpha \in \mathcal{S}$  and  $\lambda, \mu$  are boundary laminations intersecting  $\alpha$ , then we write  $d_\alpha(\lambda, \mu)$  for the diameter of the set  $\pi_\alpha(\lambda) \cup \pi_\alpha(\mu) \subset \mathcal{F}_\alpha$ . Subsurface projection is ‘locally coarsely constant’ on  $(\mathcal{BL}, \mathbf{T}_s)$  in the sense of the following lemma.

**Lemma 6.4.1.** *Let  $\gamma \in \mathcal{S}$ . For every  $\lambda \in \mathcal{BL}$  with  $\iota(\lambda, \gamma) \neq 0$  there exists a neighbourhood  $U \subset (\mathcal{BL}, \mathbf{T}_s)$  of  $\lambda$  such that for every  $\mu \in U$ ,*

$$\iota(\mu, \gamma) > 0 \text{ and } d_\gamma(\lambda, \mu) \leq 1.$$

*Proof.* Let  $\lambda$  be a boundary lamination that intersects  $\gamma$ . We will prove the lemma by assuming the opposite. Assume that there is a sequence of boundary laminations  $(\lambda_i)_{i \in \mathbb{N}}$  superconverging to  $\lambda$  such that for every  $i \in \mathbb{N}$  either  $\lambda_i$  is disjoint from  $\gamma$  or  $d_\gamma(\lambda, \lambda_i) > 1$ . Lemma 5.2.1 gives that  $\pi_\gamma(\lambda) \subset \pi_\gamma(\lambda_i)$  when  $i \in \mathbb{N}$  is sufficiently large. These  $\lambda_i$  intersect  $W_\gamma$  in disjoint arcs of at most 3 isotopy types, hence  $d_\gamma(\lambda, \lambda_i) = \text{diam } \pi_\gamma(\lambda_i) \leq 1$ . This contradicts the assumption.  $\square$

For a curve  $\gamma$  on  $\Sigma_{0,5}$  let  $\Pi_\gamma: \mathcal{BL} \rightarrow \mathbb{P}(\overline{\mathcal{F}_\gamma})$  denote the boundary projection map (Definition 5.2.2). Let  $p: I \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  be a path. For conciseness, we introduce the notation

$$\Pi_\gamma(p) = \bigcup_{t \in I} \Pi_\gamma(p(t)).$$

Observe that the fibers of boundary projection are ‘coarsely path connected’ in the sense of the following proposition.

**Proposition 6.4.2.** *There exists  $K_1 \geq 0$  with the following property. Given  $\gamma \in \mathcal{S}$ , a triangle  $\Delta \subset \mathcal{F}_\gamma$  and  $\lambda, \mu \in \mathcal{BL}$  with  $\Pi_\gamma(\lambda), \Pi_\gamma(\mu) \subset \Delta$ , there exists a path*

$p: \mathbb{I} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  from  $\mu$  to  $\lambda$  such that

$$\Pi_\gamma(p) \subset N_{K_1}(\Delta).$$

*Proof.* Fix a curve  $\gamma \in \mathcal{S}$  and a triangle  $\Delta \subset \mathcal{F}_\gamma$ . We will show that there exists a  $K_1 \geq 0$  such that any  $\lambda, \mu \in \mathcal{BL}$  with  $\Pi_\gamma(\lambda), \Pi_\gamma(\mu) \subset \Delta$  can be connected by a path that projects within a  $K_1$ -neighbourhood of  $\Delta$ . Note that for any  $\gamma' \in \mathcal{S}$  and  $\Delta' \in \mathcal{F}_{\gamma'}$  there exists a mapping class  $f \in \text{MCG}(\Sigma_{0,5})$  such that  $f(\gamma) = \gamma'$  and  $f(\Delta) = \Delta'$ . Since

$$\pi_{\gamma'} = f \circ \pi_\gamma \circ f^{-1},$$

we find that every  $\lambda, \mu \in \mathcal{BL}$  with  $\Pi_{\gamma'}(\lambda), \Pi_{\gamma'}(\mu) \subset \Delta'$  can be connected by a path that projects within a  $K_1$ -neighbourhood of  $\Delta'$ . Hence  $K_1$  is independent of the choice of  $\gamma$  and  $\Delta$ .

Choose  $\tau \in T(\gamma)$  such that  $\pi_\gamma(\tau) = \Delta$ , where  $T(\gamma)$  is the set of train tracks defined in Corollary 6.3.2. Abbreviate  $\mathbf{T} = \mathbf{T}_\gamma$ ,  $\mathbf{H} = \mathbf{H}_\gamma$  and  $\mathbf{R} = \mathbf{R}_\Delta$ . Fix a basepoint  $\kappa \in \mathcal{BL}$  that is carried by  $\tau$ . Let  $h, r: \mathbb{I} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  be paths with image disjoint from  $\mathcal{EL}(W_\gamma)$  such that  $h(0) = r(0) = \kappa$ ,  $h(1) = \mathbf{H}(\kappa)$  and  $r(1) = \mathbf{R}(\kappa)$ . Proposition 6.2.4 gives that such paths exist. Since the images of  $h$  and  $r$  are compact subsets of  $\mathcal{BL}$  that are disjoint from  $\mathcal{EL}(W_\gamma)$ , there exists  $K_1 \geq 0$  so that

$$\Pi_\gamma(h) \cup \Pi_\gamma(r) \subset N_{K_1}(\Delta),$$

by Lemma 6.4.1. Consequently,  $\kappa$  can be connected to  $f\kappa$  by a path that projects within an  $K_1$ -neighbourhood of  $\Delta$ , for any  $f \in \langle \mathbf{R}, \mathbf{H} \rangle < \text{MCG}(\Sigma_{0,5})_\gamma$ .

Suppose that  $\lambda$  is a boundary lamination with  $\Pi_\gamma(\lambda) \subset \Delta$ . We will find a path connecting  $\lambda$  to  $\kappa$  that projects near  $\Delta$ . Corollary 6.3.4 gives that  $\lambda \prec f\tau$  for some  $f \in \langle \mathbf{R}, \mathbf{H} \rangle$ . By Lemma 6.2.2.(i) and Proposition 5.4.4.(i), there is a path  $q: \mathbb{I} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  from  $\lambda$  to  $f\kappa$  with  $q(t) \prec f\tau$  for all  $t \in \mathbb{I}$ . As a consequence  $\Pi_\gamma(q) \subset \Delta$ . Concatenate  $q$  with a path from  $f\kappa$  to  $\kappa$  that projects within  $N_{K_1}(\Delta)$  to obtain a path from  $\lambda$  to  $\kappa$  that projects within  $N_{K_1}(\Delta)$ . It follows that any pair of boundary laminations  $\lambda, \mu$  with  $\Pi_\gamma(\lambda), \Pi_\gamma(\mu) \subset \Delta$  can be connected (via  $\kappa$ ) by a path  $p$  with

$$\Pi_\gamma(p) \subset N_{K_1}(\Delta). \quad \square$$

The following theorem generalises Proposition 6.4.2 and is the key result of this section.

**Theorem 6.4.3.** *There exists  $K_2 \geq 0$  with the following property. Suppose that  $\gamma \in \mathcal{S}$ ,  $\lambda, \mu \in \mathcal{BL}$  and  $g$  is a geodesic in  $\mathcal{F}_\gamma$  connecting  $x \in \Pi_\gamma(\lambda)$  and  $y \in \Pi_\gamma(\mu)$ .*

There is a path  $p: \mathbb{I} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  from  $\lambda$  to  $\mu$  such that

$$\Pi_\gamma(p) \subset N_{K_2}(g).$$

*Proof.* Let  $K_1$  be the constant from Proposition 6.4.2 and set  $K_2 = K_1 + 1$ . Suppose that  $\lambda$  and  $\mu$  are boundary laminations and  $\gamma$  is a curve on  $\Sigma_{0,5}$ . Let

$$g: [n, m] \subset \mathbb{R} \cup \{\pm\infty\} \rightarrow \mathcal{F}_\gamma$$

be a geodesic connecting some  $x \in \Pi_\gamma(\lambda)$  to  $y \in \Pi_\gamma(\mu)$ , where  $n \in \mathbb{Z} \cup \{-\infty\}$ ,  $m \in \mathbb{Z} \cup \{\infty\}$  and  $n \leq m$  (existence of  $g$  follows from Proposition 2.2.7). We may assume that  $g(t)$  is a vertex of  $\mathcal{F}_\gamma$  if and only if  $t \in [n, m] \cap \mathbb{Z}$ . For every  $i \in [n, m] \cap \mathbb{Z}$  note that  $\gamma_i = g(i)$  is a curve on  $\Sigma_{0,5}$  disjoint from  $\gamma$ . Let  $\lambda_i$  be an ending lamination on  $W_{\gamma_i}$ . Then  $\Pi_\gamma(\lambda_i) = \{\gamma_i\}$ . By Proposition 6.4.2 the laminations  $\lambda_i$  and  $\lambda_{i+1}$  can be connected by a path  $p: [i, i+1] \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  with

$$d_\gamma(p(t), g(t)) \leq K_2$$

for all  $t \in [i, i+1]$ . Concatenate these subpaths to define  $p: [n, m] \cap \mathbb{R} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  with the property  $\Pi_\gamma(p) \subset N_{K_2}(g)$ .

If  $m$  (respectively  $n$ ) is finite, we apply Proposition 6.4.2 once again to extend the domain of  $p$  over  $[m, m+1]$  (respectively  $[n-1, n]$ ), concatenating a path that projects within a  $K_2$ -neighbourhood of  $y$  (respectively  $x$ ) such that  $p(m+1) = \mu$  (respectively  $p(n-1) = \lambda$ ).

If  $m$  (respectively  $n$ ) is infinite, we define  $p(\infty) = \mu$  (respectively  $p(-\infty) = \lambda$ ). We will verify that this extends  $p$  continuously. Let  $\mathcal{N}_\mu$  denote the base for  $(\mathcal{BL}, \mathbf{T}_s)$  at  $\mu$  described in Proposition 5.2.4. It suffices to prove the following claim.

**Claim.** *For every  $N \in \mathcal{N}_\mu$  there is a  $k \in \mathbb{N}$  such that  $p(t) \in N$  for all  $t \geq k$ .*

*Proof of claim.* Suppose that  $N = N(A, \mu) \in \mathcal{N}_\mu$ . By Proposition 5.2.4 and 2.2.8, there is an edge  $e$  of  $\mathcal{F}_\gamma$  such that  $A$  is the connected component of  $\overline{\mathcal{F}_\gamma} - e$  that contains  $y$ . Passing to a smaller neighbourhood if necessary, we may assume that  $e$  separates  $x$  and  $y$ . Since  $g$  is a geodesic from  $x$  to  $y$ , there exists some  $j \in \mathbb{Z}$  such that  $g(j) \in e$ . Then for all  $t > j + K_2 + 1$

$$d_\gamma(p(t), g(t)) \leq K_2 \leq d(g(t), e),$$

implying that  $p(t) \in N(A, \mu)$ . □

As a consequence of the claim, when  $m = \infty$  (respectively  $n = -\infty$ ) we can extend  $p$  continuously to  $p(\infty) = \mu$  (respectively  $p(-\infty) = \lambda$ ). In all cases we obtain a path  $p$  from  $\lambda$  to  $\mu$  that projects within a  $K_2$ -neighbourhood of  $g$ .  $\square$

## 6.5 Local path connectivity of $(\mathcal{BL}, \mathbf{T}_s)$

In this section we will use Theorems 6.4.3 and 3.2.10 to prove that  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected (Theorem 6.5.3).

We first show local path connectivity at every boundary lamination that is not filling.

**Theorem 6.5.1.**  *$(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected at every  $\lambda \in \mathcal{BL} - \mathcal{EL}$ .*

*Proof.* Let  $\lambda$  be a boundary lamination that is disjoint from  $\gamma \in \mathcal{S}$  and fix  $x \in \partial \mathcal{F}_\gamma$  such that  $\{x\} = \Pi_\gamma(\lambda)$ . By Proposition 5.2.4 a base for the topology of superconvergence on  $\mathcal{BL}$  at  $\lambda$  is given by the sets of the form

$$N(A, \lambda) = \{\mu \in \mathcal{BL} : \Pi_\gamma(\mu) \cap A \neq \emptyset\},$$

where  $A$  ranges over the neighbourhood base  $\mathcal{A}_x$  given in Proposition 2.2.8. Hence  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected at  $\lambda$  if and only if for every  $N(A, \lambda)$  there exists  $N(A', \lambda)$  such that any  $\mu \in N(A', \lambda)$  can be connected to  $\lambda$  by a path  $p: I \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  with  $p(t) \in N(A, \lambda)$  for all  $t \in I$ . This is equivalent to the following condition.

**Claim.** *For every  $A \in \mathcal{A}_x$  there exists  $A' \in \mathcal{A}_x$  such that every  $\mu \in \mathcal{BL}$  with  $\Pi_\gamma(\mu) \subset A'$  can be connected to  $\lambda$  by a path  $p: I \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  with  $\Pi_\gamma(p) \subset A$ .*

*Proof of claim.* Let  $K_2 > 0$  be the constant from Theorem 6.4.3. Suppose that  $A \in \mathcal{A}_x$ . Let  $e$  be the edge of  $\mathcal{F}_\gamma$  such that  $A$  is a component of  $\overline{\mathcal{F}_\gamma} - e$ . Let  $e' \in E(e, x)$  be an edge in  $\mathcal{F}_\gamma$  such that  $d(e, e') > K_2$  and let  $A'$  be the component of  $\overline{\mathcal{F}_\gamma} - e'$  containing  $x$ .

Suppose that  $\mu$  is a boundary lamination with  $\Pi_\gamma(\mu) \subset A'$ . There exists a geodesic  $g$  in  $\mathcal{F}_\gamma$  from  $x$  to  $y \in \Pi_\gamma(\mu)$  (Proposition 2.2.7). By Lemma 2.2.6 the geodesic  $g$  intersects only one component of  $\overline{\mathcal{F}_\gamma} - e'$ , hence  $d_\gamma(e, g(t)) > K_2$  for all  $t$  in the domain of  $g$ . By Theorem 6.4.3 there exists a path  $p: I \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  from  $\mu$  to  $\lambda$  such that  $\Pi_\gamma(p)$  is contained in  $A$ .  $\square$

With the proof of the claim also the proof of the theorem is completed.  $\square$

The following proposition shows that  $(\mathcal{BL}, \mathbf{T}_s)$  is also locally path connected at ending laminations. Its proof relies on Gabai's result that  $(\mathcal{EL}, \mathbf{T}_s)$  is locally path connected (Theorem 3.2.10).

**Proposition 6.5.2.**  *$(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected at every  $\lambda \in \mathcal{EL}$ .*

*Proof.* Write  $X = (\mathcal{BL}, \mathbf{T}_s)$  and  $X' = (\mathcal{EL}, \mathbf{T}_s)$ . By definition of the topology of superconvergence (Definition 3.2.4) and Lemma 6.1.1, the space  $X'$  is a dense subspace of  $X$ . Suppose that  $\lambda \in X'$ . Let  $N$  be a neighbourhood of  $\lambda$  in  $X$ . Take an open set  $U \subset X$  so that  $\lambda \in U \subset N$ . Since open sets in  $X'$  are exactly the restrictions of open sets in  $X$  to  $X'$ , the set  $U' = U \cap X'$  is an open neighbourhood of  $\lambda$  in  $X'$ . By Theorem 3.2.10  $X'$  is locally path connected at  $\lambda$ , so there exists a neighbourhood  $M' \subset X'$  of  $\lambda$  that has the following property. For every  $\mu \in M'$  there is a path  $p: I \rightarrow X'$  with  $p(0) = \mu$ ,  $p(1) = \lambda$  and  $p(t) \in U'$  for all  $0 \leq t \leq 1$ . Choose an open set  $V' \subset X'$  with  $\lambda \in V' \subset M'$ . Let  $V \subset X$  be an open set such that  $V' = V \cap X'$  and  $V \subset U$ .

We will show that every  $\mu \in V$  can be connected to  $\lambda$  by a path in  $N$ . If  $\mu \in V$  is an ending lamination then  $\mu \in M'$ , hence there is a path in  $U'$  from  $\lambda$  to  $\mu$  which includes into  $X$  to give the desired path. Now suppose that  $\mu \in V$  fills a four-times punctured sphere. It suffices to observe that  $\mu$  can be connected to an ending lamination by a path in  $V$ . This is immediate from the facts that  $V$  is open,  $X$  is locally path connected at  $\mu$  (Proposition 6.5.1) and  $X' \subset X$  is dense (Lemma 6.1.1).  $\square$

From Theorem 6.5.1 and Proposition 6.5.2 we deduce the following.

**Theorem 6.5.3.**  *$(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected.*

*Proof.* By Theorem 6.5.1  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected at laminations that fill a four-holed sphere and by Proposition 6.5.2  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected at ending laminations. So  $(\mathcal{BL}, \mathbf{T}_s)$  is locally path connected at every point.  $\square$

In particular, combining observations from this and the previous chapter, we draw the following conclusion.

**Corollary 6.5.4.**  *$\partial\mathcal{P}$  is path connected and locally path connected.*

*Proof.* Corollary 5.3.5 gives that  $\partial\mathcal{P} \cong (\mathcal{BL}, \mathbf{T}_s)$ . The result then follows from Proposition 6.2.3 and Theorem 6.5.3.  $\square$

## Chapter 7

# Dimensions of lamination spaces

In this chapter we investigate the dimensions of  $\mathcal{PABL}$  and of the space of boundary laminations with the topology of superconvergence. This part of the thesis is fairly open-ended and gives some directions for future research.

Explicitly, we discuss the following observations and open questions. First of all, we observe that  $\mathcal{PABL}$  is 1- or 2-dimensional (Lemma 7.2.2). We use an established theorem of Mazurkiewicz (Theorem 7.1.5) to relate the dimension of  $\mathcal{PABL}$  to the connectivity of the curve graph inside  $\mathcal{PML}(\Sigma_{0,5})$ . This leads us to ask Question 7.2.3.

Secondly, we observe that  $(\mathcal{BL}, \mathbf{T}_s)$  has dimension 1 or 2 (Lemma 7.3.2). In an attempt to improve this result we analyse the neighbourhoods at non-ending laminations in  $(\mathcal{BL}, \mathbf{T}_s)$  provided by Proposition 5.2.4. We show that the frontiers of these neighbourhoods contain paths (Proposition 7.3.6). Question 7.3.3 is posed as a starting point for further investigation.

### 7.1 Topological dimension

This section gives an overview of the preliminary definitions and results that will be used in the rest of the chapter. A standard reference for this material is Engelking's book on dimension theory [Eng1].

Recall that a topological space is *separable* if it has a countable dense subset. Throughout the section, we assume that  $X$  is a separable and metrisable space. For extensions to larger classes of spaces we refer the reader to [Eng1].

**Definition 7.1.1** (Dimension). We define the (*small inductive*) *dimension* of  $X$ , denoted  $\dim X \in \mathbb{Z}$ , as follows.

- $\dim X = -1$  if and only if  $X = \emptyset$ .

- $\dim X \leq k$  if and only if for every  $x \in X$  and every neighbourhood  $N \subset X$  of  $x$  there exists an open set  $U \subset N$  such that

$$x \in U \subset N \quad \text{and} \quad \dim(\text{fr } U) \leq k - 1.$$

If  $X$  does not have this property we write  $\dim X > k$ .

- $\dim X = k$  if  $\dim X \leq k$  and  $\dim X > k - 1$ .

*Remark 7.1.2.* It is common to write  $\text{ind } X$  for the small inductive dimension of  $X$ , reserving the notation  $\dim X$  for the covering dimension. However, since we are only concerned with separable metrisable spaces, these two notions of dimension coincide [Eng1, 1.7.7]. This motivates our notation  $\dim X$ , called the *dimension* of  $X$  for short.

Homeomorphic spaces have the same dimension, in other words, the dimension is a topological invariant. We highlight the following properties, proved in [Eng1, 1.1.2, 1.5.2, 1.5.3 and 1.8.10 (respectively)].

**Lemma 7.1.3.**

- (i) For every subspace  $Y$  of  $X$  we have  $\dim Y \leq \dim X$ .
- (ii) If  $Y, Z \subset X$  such that  $\dim Y \leq n - 1$ ,  $\dim Z \leq 0$  and  $X = Y \cup Z$ , then  $\dim X \leq n$ .
- (iii) Suppose that for every  $i \in \mathbb{N}$ ,  $Y_i$  is a closed subset of  $X$  with  $\dim Y_i \leq n$ . If  $X = \bigcup_{i \in \mathbb{N}} Y_i$ , then  $\dim X \leq n$ .
- (iv) A subspace  $U$  of  $\mathbb{R}^n$  satisfies  $\dim U = n$  if and only if it has non-empty interior.

Let  $\mathcal{N}_m^n$  be the subspace of  $\mathbb{R}^n$  consisting of all points with at most  $m$  rational coordinates. Then  $\dim \mathcal{N}_m^n = m$  and  $\dim(\mathbb{R}^n - \mathcal{N}_m^n) = n - m - 1$  [Eng1, 1.8.5]. Nöbeling proved that the space  $\mathcal{N}_m^{2m+1}$  is ‘universal’ in the sense that it contains every separable metrisable space of dimension at most  $m$  as a subspace [Nöb].

**Definition 7.1.4.**  $\mathcal{N}_m^{2m+1} \subset \mathbb{R}^{2m+1}$  is called the *m-dimensional Nöbeling space*. The 1-dimensional Nöbeling space is called the *Nöbeling curve*.

Recall that a *continuum* is a connected, compact and metrisable space. In 1929, Mazurkiewicz proved the following theorem about codimension-2 subsets of  $\mathbb{R}^n$  [Maz, Théorème 2]. For a more recent account, see also [Eng1, 1.8.19].

**Theorem 7.1.5** (Mazurkiewicz). *Let  $U$  be a connected open set in  $\mathbb{R}^n$ . If  $A$  is a subset of  $U$  with  $\dim A \leq n - 2$ , then for every  $x, y \in U - A$  there exist a continuum  $X \subset U - A$  which contains  $x$  and  $y$ .*

We use Mazurkiewicz's theorem to prove the following corollary.

**Corollary 7.1.6.** *If  $A \subset \mathbb{S}^n$  such that  $\dim A \leq n - 2$ , then  $\mathbb{S}^n - A$  is connected and locally connected.*

*Proof.* Suppose that  $A \subset \mathbb{S}^n$  has dimension at most  $n - 2$ . We may assume that  $A$  is non-empty. Let  $U \subset \mathbb{S}^n$  be a proper subset that is connected and open. By Lemma 7.1.3.(i),  $\dim(U \cap A) \leq n - 2$ . Theorem 7.1.5 implies that  $U - A$  is connected. Since this holds for any  $U$ , this shows that  $\mathbb{S}^n - A$  is locally connected.

Taking  $U = \mathbb{S}^n - \{a\}$  for some  $a \in A$ , the same argument shows that  $U - A = \mathbb{S}^n - A$  is connected.  $\square$

## 7.2 The dimension of $\mathcal{PABL}$

Write  $\mathcal{C}$  for the curve graph of  $\Sigma_{0,5}$  and let  $\eta: \mathcal{C} \rightarrow \mathcal{PML}(\Sigma_{0,5})$  be the injection described in Definition 4.1.1. As subspaces of a separable metric space,  $\eta(\mathcal{C})$  and  $\mathcal{PABL}$  are separable and metrisable themselves.

**Lemma 7.2.1.**  $\dim \eta(\mathcal{C}) = 1$ .

*Proof.* Recall that  $\eta(\mathcal{C})$  is the union of countably many PL intervals  $D_{\alpha,\beta}$ , where  $\alpha$  and  $\beta$  are distinct curves on  $\Sigma_{0,5}$  (page 47). Each  $D_{\alpha,\beta}$  is a closed and 1-dimensional set, hence  $\dim \eta(\mathcal{C}) \leq 1$  by Lemma 7.1.3.(iii).  $\square$

**Lemma 7.2.2.**  $1 \leq \dim \mathcal{PABL} \leq 2$ .

*Proof.* By Lemma 6.2.2.(iii) there exists a non-constant path in  $\mathcal{PABL}$ , hence  $\mathcal{PABL}$  is not 0-dimensional. On the other hand, since curves are dense in  $\mathcal{PML}(\Sigma_{0,5})$  (Remark 3.2.22),  $\mathcal{PABL}$  is the complement of a dense set in  $\mathcal{PML}(\Sigma_{0,5}) \cong \mathbb{S}^3$ . Lemma 7.1.3.(iv) gives that  $\dim \mathcal{PABL}$  is at most 2.  $\square$

To get more insight in the dimension of  $\mathcal{PABL}$ , we ask the following related question.

**Question 7.2.3.** Is  $\eta(\mathcal{C}) \subset \mathcal{PML}(\Sigma_{0,5})$  locally connected?

If the answer to Question 7.2.3 is negative, then it follows from Corollary 7.1.6 that  $\mathcal{PABL}$  is 2-dimensional. However, it is not so clear whether or not  $\eta(\mathcal{C})$  is locally connected. There exist paths in  $\eta(\mathcal{C})$  that do not come from paths in  $\mathcal{C}$ , as the following example demonstrates.



*Example 7.2.4.* Suppose that  $\alpha, \beta \in \mathcal{S}(\Sigma_{0,5})$  are intersecting curves. Let  $p: I \rightarrow \mathcal{C}$  be a path from  $\alpha$  to  $T_\beta(\alpha)$  that is a finite concatenation of edges. Then  $q_i = \eta T_\beta^i p$  is a path in  $\mathcal{PML}(\Sigma_{0,5})$  from  $T_\beta^i(\alpha)$  to  $T_\beta^{i+1}(\alpha)$ . For every neighbourhood  $N \subset \mathcal{PML}(\Sigma_{0,5})$  of  $\beta$  there exists  $j \in \mathbb{N}$  such that  $q_i \subset N$  for all  $i \geq j$ . Therefore, the concatenation

$$q: [0, \infty] \rightarrow \mathcal{PML}(\Sigma_{0,5})$$

with the property that  $q|_{[i, i+1]} = q_i$  and  $q(\infty) = \beta$  is a path from  $\alpha$  to  $\beta$  in  $\eta(\mathcal{C})$ . However,  $\eta^{-1} \circ q$  is discontinuous with respect to the metric and CW topologies on  $\mathcal{C}$ . Compare Remark 4.1.2.

On a side note, observe that there exist locally path connected ‘graphs’ in  $\mathbb{R}^3$  whose complement has dimension 1. See the following example.

*Example 7.2.5.* We describe a countable graph  $\Gamma$  and a continuous injection  $J: \Gamma \rightarrow \mathbb{R}^3$  that maps the edges of  $\Gamma$  to straight line segments. Furthermore,  $J(\Gamma)$  is locally path connected and its complement is 1-dimensional.

For  $n = 2, 3$ , let  $\mathcal{D}(n) \subset \mathbb{R}^n$  denote the subset consisting of all points with at least  $n - 1$  dyadic rational coordinates. It is not hard to see that  $\mathcal{D}(n)$  is locally path connected. There exists a homeomorphism  $I \rightarrow I$  that identifies the dyadic rationals with the rationals (for instance, the Minkowski question-mark function [Mink]). Consequently, there exists a homeomorphism  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  that identifies  $\mathcal{D}(3)$  with the complement of the Nöbeling curve. In particular, both  $\mathcal{D}(3)$  and its complement are 1-dimensional. Hence it suffices to find a graph  $\Gamma$  and an injection  $J$  such that  $J(\Gamma) = \mathcal{D}(n)$ .

We discuss the case  $n = 2$  as a model for the case  $n = 3$ . Let  $A_m$  be the set of all  $(x, y) \in \mathbb{R}^2$  for which at least one of  $2^m x$  and  $2^m y$  is an integer. Note that  $A_0$  is the integer grid,  $A_m \subset A_{m+1}$  and  $\bigcup_{m \in \mathbb{N}} A_m = \mathcal{D}(2)$ . Clearly  $A_0$  is homeomorphic to a 4-regular graph  $\Gamma_0$ . Let  $T_k$  denote the countable tree that has one vertex of valence  $k$  and all other vertices are 2-valent. For every  $m \in \mathbb{N}$ , every connected component of  $A_{m+1} - A_m$  is homeomorphic to  $T_4$ . For every  $i \in \mathbb{N}$  define  $\Gamma_i = T_4$ . There is a continuous bijection  $J$  from the (disconnected) graph

$$\Gamma = \bigsqcup_{i=0}^{\infty} \Gamma_i$$

to  $\mathcal{D}(2) \subset \mathbb{R}^2$  that restricts to an embedding of every  $\Gamma_i$ , illustrated in Figure 7.1.

This construction can be adapted to ensure that  $\Gamma$  is connected, compare Figure 7.2. Roughly, for every  $i \in \mathbb{N}$  we replace one ray of edges in  $\Gamma_i$  by a single edge, attaching it to some  $\Gamma_j$  (possibly subdividing an edge thereof). Choosing

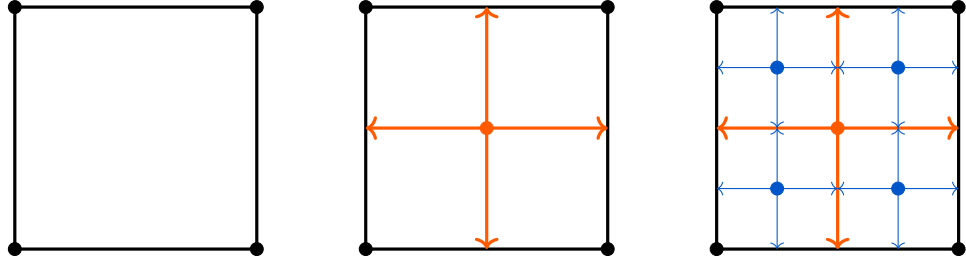


Figure 7.1:  $\mathcal{D}(1)$  is the union of the integer grid and countably many copies of  $T_4$ . Here a dot indicates a vertex, a segment connecting two dots indicates an edge and a segment that ends in an arrowhead indicates an infinite ray of edges connected by valence-2 vertices.

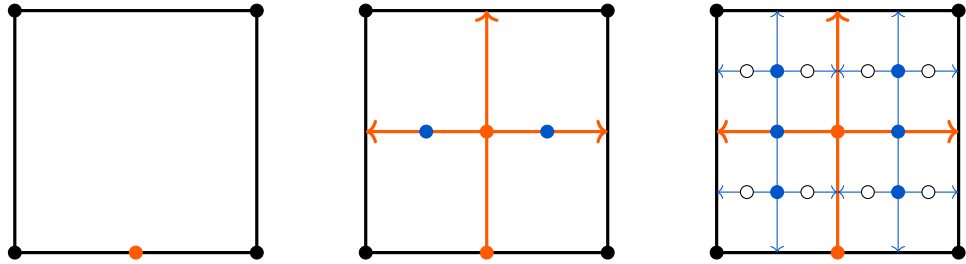


Figure 7.2:  $\mathcal{D}(1)$  is the image of a connected graph with vertices of valence  $\leq 4$ .

carefully where to attach, we can ensure that no edge gets subdivided infinitely often.

Now suppose that  $n = 3$ . Similar to situation explained above,  $\mathcal{D}(3)$  is the union of the integer grid in  $\mathbb{R}^3$  and countably many copies of  $T_4$  and  $T_6$ . This can again be adapted to find a connected graph with vertices of valence at most 6 that continuously bijects to  $\mathcal{D}(3)$ .

### 7.3 The dimension of $(\mathcal{BL}, \mathbf{T}_s)$

In [HenP, §5], Hensel and Przytycki show that the ending lamination space of the five-times punctured sphere is 1-dimensional. In fact, they find the following topological characterisation [HenP, Theorem 1.3].

**Theorem 7.3.1** (Hensel-Przytycki).  *$\mathcal{EL}(\Sigma_{0,5})$  is homeomorphic to Nöbeling curve.*

We remark that  $(\mathcal{BL}, \mathbf{T}_s)$  is metrisable and separable. Since  $(\mathcal{BL}, \mathbf{T}_s) \cong \partial \mathcal{P}$  (Corollary 5.3.5) and Gromov boundaries are metrisable (see Section 2.1), it follows that  $(\mathcal{BL}, \mathbf{T}_s)$  is metrisable. There is a continuous surjection  $\mathcal{PABL} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  (Proposition 5.4.4.(i)) and  $\mathcal{PABL}$  is separable (compare Remark 5.4.3), which implies that  $(\mathcal{BL}, \mathbf{T}_s)$  is separable.

Using that  $\mathcal{EL}(\Sigma_{0,5})$  has dimension 1, we prove the following lemma.

**Lemma 7.3.2.**  $1 \leq \dim(\mathcal{BL}, \mathbf{T}_s) \leq 2$ .

*Proof.* Since  $(\mathcal{BL}, \mathbf{T}_s)$  is path connected (Proposition 6.2.3) its dimension is at least 1. For every  $\gamma \in \mathcal{S}(\Sigma_{0,5})$  the set  $\mathcal{EL}(W_\gamma) \cong \mathbb{R} - \mathbb{Q}$  is a closed 0-dimensional subset of  $(\mathcal{BL}, \mathbf{T}_s)$ . By Lemma 7.1.3.(iii) the countable union

$$\bigcup_{\gamma \in \mathcal{S}(\Sigma_{0,5})} \mathcal{EL}(W_\gamma) \subset (\mathcal{BL}, \mathbf{T}_s)$$

has dimension 0. By Theorem 7.3.1 the dimension of  $\mathcal{EL}(\Sigma_{0,5})$  is 1. Being the union of a 0-dimensional and a 1-dimensional subspace,  $(\mathcal{BL}, \mathbf{T}_s)$  is at most 2-dimensional (Lemma 7.1.3.(ii)).  $\square$

We ask the following question.

*Question 7.3.3.* Is  $(\mathcal{BL}, \mathbf{T}_s)$  1-dimensional?

In an attempt to answer this question, we study the neighbourhood base  $\mathcal{N}_\lambda$  of  $(\mathcal{BL}, \mathbf{T}_s)$  found in Proposition 5.2.4. We will show that the frontiers of the neighbourhoods in  $\mathcal{N}_\lambda$  are not 0-dimensional (Proposition 7.3.6). The main obstruction is captured in the following lemma. Recall that  $\Pi_\gamma: \mathcal{BL} \rightarrow \mathbb{P}(\overline{\mathcal{F}_\gamma})$  is the boundary projection map (Definition 5.2.2).

**Lemma 7.3.4.** *Suppose that  $\gamma \in \mathcal{S}(\Sigma_{0,5})$  and  $e$  is an edge of  $\mathcal{F}_\gamma$ . There exists a non-constant path  $p: I \rightarrow (\mathcal{BL}, \mathbf{T}_s)$  such that  $\Pi_\gamma(p) \subset e$ .*

We prove this lemma in two ways. A sketch of a proof that relies on the theory of measured foliations is given in Remark 7.3.5. However, since we have not introduced this setup, we also include an explicit proof. This explicit proof only uses train tracks and laminations.

*Remark 7.3.5.* The reader familiar with the theory of measured foliations might prove Lemma 7.3.4 as follows. Take a minimal lamination  $\lambda$  on  $\Sigma_{0,5}$  that fills a four-holed sphere disjoint from a curve  $\alpha \in e \subset \mathcal{F}_\gamma$  and extend it to a singular foliation  $F$ . We may assume that this foliation has no closed leaves and has singularities only at punctures, which will be interpreted as ‘marked points’. Exactly one of the marked points is an interior point of a singular leaf. Moving this marked point continuously across the leaves of  $\lambda$ , we obtain a new foliation  $F'$  of  $\Sigma_{0,5}$ . Figure 7.3 illustrates this procedure. Since  $F'$  again has no closed leaves, the corresponding lamination  $\lambda'$  is minimal. Furthermore,  $\lambda'$  is filling (respectively fills a four-holed sphere) if and only if the moved marked point is contained in a regular (respectively singular) leaf of  $F'$ .

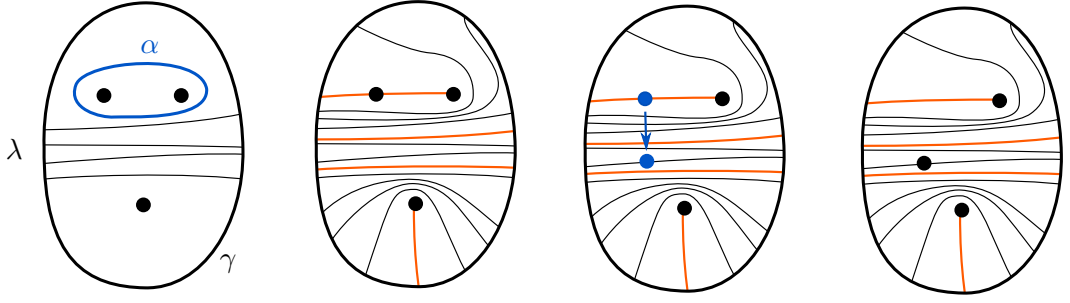


Figure 7.3: Moving the puncture across the leaves of  $\lambda$  gives a path of foliations on  $\Sigma_{0,5}$ .

This procedure is an elementary case of a more general procedure described in [LeiS] and subsequently in [ChH]. See [Lev] for a discussion of the correspondence between laminations and foliations.

*Proof of Lemma 7.3.4.* If the lemma holds for one particular curve and one particular edge, then we can apply the mapping class group to see it holds for any. Let  $\tau$  be the birecurrent train track and  $\gamma$  the curve on  $\Sigma_{0,5}$  displayed in Figure 7.4. Let  $e$  be the edge  $\pi_\gamma(\tau) \subset \mathcal{F}_\gamma$ . We will show that the polytope  $P(\tau)$  contains a path in  $\mathcal{PBL}$  which connects two non-filling laminations. Composed with the map  $\Omega$  defined in Proposition 5.4.4.(i) this will give a non-constant path in  $(\mathcal{BL}, \mathbf{T}_s)$ .

Let  $b_1, b_2, b_3, b_4$  denote the branches of  $\tau$ , labelled as in Figure 7.4. Write  $w_k \in \mathbb{R}_{\geq 0}$  for the weight of a lamination carried by  $\tau$  on the branch  $b_k$ , so that

$$P(\tau) = \{[w_1 : w_2 : w_3 : w_4] \in \mathbb{R}_{\geq 0}^4 : w_1 + w_2 = w_3 + w_4\}.$$

We identify  $P(\tau)$  with the unit square via the homeomorphism

$$\Lambda : \mathbb{I}^2 \rightarrow P(\tau), \quad (x, y) \mapsto [x : 1 - x : y : 1 - y].$$

In what follows, we use the correspondence  $\Lambda$  to assign coordinates in  $\mathbb{I}^2$  to elements of  $P(\tau)$ .

The main part of the proof is subdivided into two claims. The first claim is used to prove the second. The second claim implies the lemma.

**Claim.** *If  $(x_0, y_0), (x_1, y_1) \in (\mathbb{Q} \cap \mathbb{I}) \times \mathbb{I}$  with  $x_0 \neq x_1$ , then  $\Lambda(x_0, y_0)$  and  $\Lambda(x_1, y_1)$  are multicurves that do not have a common curve.*

*Proof of claim.* Observe that for  $x \in \{0, 1\}$ , the vertex cycles  $\Lambda(x, 0)$  and  $\Lambda(x, 1)$  are disjoint curves. It follows that  $\{\Lambda(x, y) : x \in \{0, 1\}\} \subset \eta(\mathcal{C})$ .

A similar result holds for any rational  $x$ -coordinate. Let  $m > n > 0$  be coprime integers such that  $x = \frac{n}{m}$ . Since  $n$  and  $m$  are coprime,  $\Lambda(\frac{n}{m}, 0) = [n : m - n : 0 : m]$  is a curve. Furthermore, if  $0 \leq k \leq m - 1$  then  $\Lambda(\frac{n}{m}, \frac{k}{m})$  and  $\Lambda(\frac{n}{m}, \frac{k+1}{m})$  are disjoint curves. This follows from a simple curve surgery, in which one arc ‘overtakes’ one puncture. We conclude that if  $\frac{k}{m} < y < \frac{k+1}{m}$  then  $\Lambda(x, y)$  is the multicurve  $\Lambda(x, \frac{k}{m}) \cup \Lambda(x, \frac{k+1}{m})$ . This is illustrated in Figure 7.5.  $\square$

In particular, the first claim implies that  $\{\Lambda(x, y) : x \in \mathbb{Q}\} \subset \eta(\mathcal{C})$ .

**Claim.**  $\mathcal{PBL}$  intersects  $P(\tau)$  in the set  $\{\Lambda(x, y) : x \in \mathbb{R} - \mathbb{Q}\}$ .

*Proof of claim.* Define  $Z = \{(x, y) \in \mathbb{I}^2 : x \in \mathbb{R} - \mathbb{Q}\}$ . We have already shown that  $P(\tau) - \Lambda(Z) \subset \eta(\mathcal{C})$ , which implies that  $\mathcal{PBL} \cap P(\tau) \subset \Lambda(Z)$ . It remains to show that  $\Lambda(Z) \subset \mathcal{PBL}$ .

Suppose that  $(x_0, y_0) \in Z$ . Since  $x_0$  is irrational, also  $(1 - x_0)/x_0$  is irrational. In particular,  $\lambda = \Lambda(x_0, y_0)$  cannot be a curve. We will prove that  $\lambda$  is a minimal lamination by assuming the opposite. If  $\lambda$  is non-minimal, then  $\lambda = s\alpha + (1 - s)\mu$  for some curve  $\alpha \in \mathcal{S}(\Sigma_{0,5})$  and a minimal lamination  $\mu$ . Since  $\alpha$  is a sublamination of  $\lambda$ , it is carried by  $\tau$ . Define  $(x_1, y_1) = \Lambda^{-1}(\alpha)$  and note that  $x_1 \in \mathbb{Q}$ . Consider the path

$$q: \mathbb{I} \rightarrow \mathbb{I}^2, \quad q(t) = t(x_1, y_1) + (1 - t)(x_0, y_0).$$

Since  $x_1 \in \mathbb{Q}$  and  $x_0 \in \mathbb{R} - \mathbb{Q}$ , the path  $q$  passes through countably many points with distinct rational  $x$ -coordinates. The first claim then gives that  $\Lambda \circ q$  passes through countably many distinct multicurves.

On the other hand, every point on  $\Lambda \circ q$  is a weighted sum of  $\alpha$  and  $\mu$ . This is a contradiction, hence  $\lambda$  must be minimal. We conclude that  $\lambda \in \mathcal{PBL}$ .  $\square$

By the second claim, for any  $r \in \mathbb{I} - \mathbb{Q}$  the image of the path

$$p: \mathbb{I} \rightarrow P(\tau), \quad p(t) = \Lambda(r, t),$$

is contained in  $\mathcal{PBL}$ . Postcomposing  $p$  with the continuous map  $\Omega$  defined in Proposition 5.4.4.(i), we obtain a path  $\Omega \circ p: \mathbb{I} \rightarrow (\mathcal{BL}, \mathbf{T}_s)$ . For all  $t \in \mathbb{I}$  the lamination  $\Omega \circ p(t)$  is carried by  $\tau$ . Observe that  $\Pi_\gamma(\lambda) \subset e$  for any  $\lambda \prec \tau$ . Since the endpoints of  $p$  are laminations filling distinct four-holed spheres,  $\Omega \circ p$  is non-constant.  $\square$

Lemma 7.3.4 implies that the frontiers of our candidate neighbourhoods (provided by Proposition 5.2.4) contain paths.

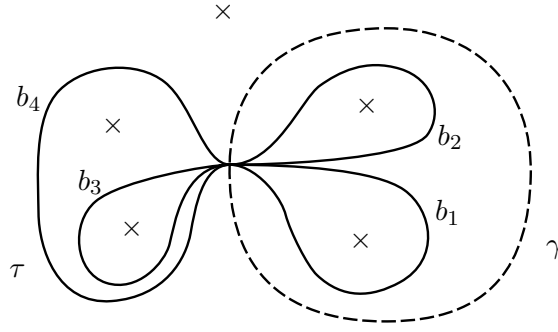


Figure 7.4: The train track  $\tau$  and the curve  $\gamma$  on  $\Sigma_{0,5}$ .

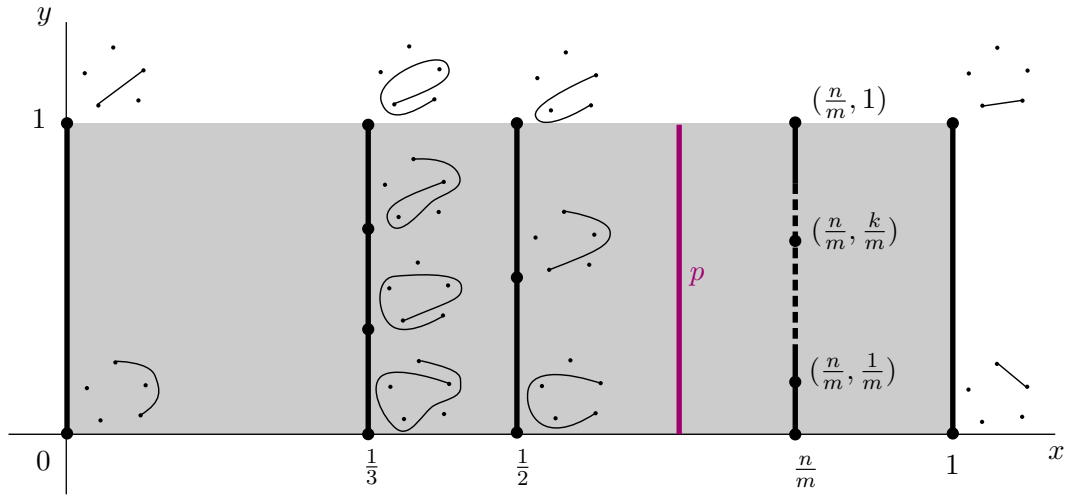


Figure 7.5: The polygon  $P(\tau)$ , with coordinates given by the correspondence  $\Lambda$ . Here a black dot indicates a point  $(\frac{n}{m}, \frac{k}{m}) \in \mathbb{I}^2$ , and the picture next to it represents the curve  $\Lambda(\frac{n}{m}, \frac{k}{m})$ . A segment connecting two black dots corresponds to an edge of  $\eta(\mathcal{C})$ .

**Proposition 7.3.6.** *Let  $\lambda \in \mathcal{BL}$  be a boundary lamination that is not filling and let  $\mathcal{N}_\lambda$  be the open neighbourhood base at  $\lambda$  given by Proposition 5.2.4. For any  $N \in \mathcal{N}_\lambda$*

$$\dim(\text{fr } N) > 0.$$

*Proof.* Let  $\lambda \in \mathcal{BL}$  be a boundary lamination disjoint from  $\gamma \in \mathcal{S}(\Sigma)$  and take  $x \in \partial \mathcal{F}_\gamma$  such that  $\{x\} = \Pi_\gamma(\lambda)$ . Let  $N \in \mathcal{N}_\lambda$ . By the definitions of  $\mathcal{N}_\lambda$  (Proposition 5.2.4) and  $\mathcal{A}_x$  (Proposition 2.2.8), there exists an edge  $e \in E(\mathcal{F}_\gamma)$  such that if  $A$  is the connected component of  $\overline{\mathcal{F}_\gamma} - e$  containing  $x$ , then

$$N = \{\mu \in \mathcal{BL} : \Pi_\gamma(\mu) \cap A \neq \emptyset\}.$$

By Proposition 5.2.4 the set  $N$  is open, and the set

$$C = \{\mu \in \mathcal{BL} : \Pi_\gamma(\mu) \subset A \cup e\}$$

is closed in  $(\mathcal{BL}, \mathbf{T}_s)$ . Since the diameter of every  $\Pi_\gamma(\mu)$  is at most 1, we have that  $N \subset C$ . We claim that  $C$  is actually the smallest closed set that contains  $N$ .

**Claim.** *For every  $\mu \in C$  there exist  $\mu_i \in N$  such that  $(\mu_i)_{i \in \mathbb{N}}$  superconverges to  $\mu$ .*

*Proof of claim.* If  $\mu \in N$  we can just take  $\mu_i = \mu$  for every  $i \in \mathbb{N}$ . Assume that  $\mu \in C - N$ , or equivalently  $\Pi_\gamma(\mu) \subset e$ . Let  $\Delta$  be the triangle in  $\mathcal{F}_\gamma$  with

$$e \subset \Delta \subset A \cup e.$$

By Theorem 6.3.3(ii), there exists a complete train track  $\tau$  such that  $\mu \prec \tau$  and  $\pi_\gamma(\tau) = \Delta$ . Being the complement of countably many intervals and points,  $\mathcal{PABL}$  is dense in  $P(\tau)$  (Lemma B.1). So there exist laminations  $\mu_i \in \mathcal{PABL}$  fully carried by  $\tau$  such that  $(\mu_i)_{i \in \mathbb{N}}$  superconverges to  $\tau$ . Then  $\Pi_\gamma(\mu_i) = \Delta$ , hence  $\mu_i \in N$  for every  $i \in \mathbb{N}$ .  $\square$

It follows that  $C$  is the closure of  $N$ . Therefore, the frontier of  $N$  is the set

$$F = \{\mu \in \mathcal{BL} : \Pi_\gamma(\mu) \subset e\}.$$

Lemma 7.3.4 shows that  $F$  contains a non-constant path, hence  $\dim F > 0$ .  $\square$

Question 7.3.3 remains open and could be a starting point for future research.

# Appendix A

## Topological notions

We recall some concepts from point-set topology.

### Neighbourhood bases

Bases, neighbourhoods and neighbourhood bases are widely known concepts in point-set topology. However, different authors appear to have different conventions (for instance, some authors require neighbourhoods to be open). In this section we set up our conventions, following Willard's book [Wil].

**Definition A.1** (Neighbourhood base). Let  $X$  be a topological space and suppose that  $x \in X$ .

- $N \subset X$  is a *neighbourhood of  $x$*  if there exists an open set  $U \subset X$  such that  $x \in U \subset N$ .
- A *neighbourhood base at  $x$*  is a collection  $\mathcal{N}_x$  of neighbourhoods of  $x$ , such that any neighbourhood  $M$  of  $x$  contains some  $N \in \mathcal{N}_x$ .
- A neighbourhood base at  $x$  is *open* if it consists only of open sets.

Neighbourhood bases describe topologies, in the sense of the following theorem [Wil, Theorem 4.5].

**Theorem A.2.** Let  $X$  be a topological space and for every  $x \in X$  let  $\mathcal{N}_x$  be a neighbourhood base at  $x$ . The following holds.

(V1) If  $N \in \mathcal{N}_x$  then  $x \in N$ .

(V2) If  $N, M \in \mathcal{N}_x$  then there is some  $O \in \mathcal{N}_x$  such that  $O \subset N \cap M$ .



(V3) If  $N \in \mathcal{N}_x$  then there is some  $M \in \mathcal{N}_x$  such that for any  $y \in M$  there is some  $O \in \mathcal{N}_y$  with  $O \subset N$ .

(V4)  $U \subset X$  is open  $\iff$  for every  $x \in U$  there is an  $N \in \mathcal{N}_x$  such that  $N \subset U$ .

Conversely, if  $X$  is a set and for every  $x \in X$  we have a collection  $\mathcal{N}_x$  of subsets of  $X$  satisfying (V1)–(V3), then (V4) defines a topology on  $X$ . In this topology  $\mathcal{N}_x$  is a neighbourhood base at  $x$ , for each  $x \in X$ .

A perhaps more familiar notion is a base for a topology.

**Definition A.3** (Base). If  $X$  is a topological space, a *base* for  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  such that  $U \subset X$  is open if and only if  $U$  is a union of elements in  $\mathcal{B}$ .

Bases relate to neighbourhood bases in the following way [Wil, Theorem 5.4].

**Theorem A.4.** Let  $X$  be a topological space and let  $\mathcal{B}$  be a collection of open sets in  $X$ . Then  $\mathcal{B}$  is a base for  $X$  if and only if for every  $x \in X$  the collection  $\{B \in \mathcal{B} : x \in B\}$  is a neighbourhood base at  $x$ .

## Topologies described by sequences

In this section we discuss how a particular ‘notion of convergence’ can be used to generate a topology on a set. A reference for this topic is, for instance, Engelking’s book [Eng2]. See also the original papers by Fréchet [Fré1, Fré2] and Urysohn [Ury].

**Definition A.5** ( $\mathcal{S}^*$ -space). Let  $X$  be a set and let  $\mathfrak{S}X$  denote the collection of sequences of points in  $X$ . A *limit operator*  $L$  on  $X$  is a map

$$L: C \subset \mathfrak{S}X \rightarrow X.$$

We say that a sequence  $s \in \mathfrak{S}X$   *$L$ -converges* to  $x \in X$  if  $s \in C$  and  $L(s) = x$ . Assume that  $L$  and  $C$  satisfy the following conditions.

- (L1) If  $s = (x)_{i \in \mathbb{N}} \in \mathfrak{S}X$  is a constant sequence, then  $s \in C$  and  $L(s) = x$ .
- (L2) If  $s \in \mathfrak{S}X$   $L$ -converges to  $x$ , then every subsequence of  $s$   $L$ -converges to  $x$ .
- (L3) If  $s \in \mathfrak{S}X$  does not  $L$ -converge to  $x$ , then  $s$  has a subsequence such that no further subsequence  $L$ -converges to  $x$ .
- (L4) If  $(x_i)_{i \in \mathbb{N}} \in \mathfrak{S}X$   $L$ -converges to  $x$  and  $(x_j^i)_{j \in \mathbb{N}} \in \mathfrak{S}X$   $L$ -converges to  $x_i$  for any  $i \in \mathbb{N}$ , then there exist sequences of natural numbers  $i(1), i(2), \dots$  and  $j(1), j(2), \dots$  such that  $(x_{j(k)}^{i(k)})_{k \in \mathbb{N}}$   $L$ -converges to  $x$ .

The data  $(X, L)$  is called an  $\mathcal{S}^*$ -space.

Say that a subset  $K \subset X$  is  $L$ -closed if and only if for every sequence  $(x_i)_{i \in \mathbb{N}}$  of points in  $K$  that  $L$ -converges  $x$ , we have that  $x \in K$ . These sets generate a topology on  $X$  [Eng2, 1.7.18–19].

**Definition A.6.** Given an  $\mathcal{S}^*$ -space  $(X, L)$ , the *Fréchet topology* on  $X$  induced by  $L$  is the topology whose closed sets are the  $L$ -closed sets.

Recall that a topological space  $X$  is *sequential* if every sequentially closed set is closed. Furthermore, we say that  $X$  is a *Fréchet space* if for every subset  $A$  of  $X$  the *sequential closure* of  $A$  in  $X$

$$\{x \in X : \text{there exists } s \in \mathfrak{S}A \text{ that converges to } x\}$$

equals the closure of  $A$  in  $X$ . Note that  $X$  is a Fréchet space if and only if every subspace of  $X$  is sequential.

The Fréchet topology has the following properties [Eng2, 1.7.18.b and c, 1.7.20].

**Lemma A.7.** *Let  $(X, L)$  be an  $\mathcal{S}^*$ -space.*

- (i)  *$X$  with the Fréchet topology is a Fréchet space.*
- (ii)  *$(x_n)_{n \in \mathbb{N}} \in \mathfrak{S}X$   $L$ -converges to  $x$  if and only if  $x_n \rightarrow x$  in the Fréchet topology.*
- (iii) *Suppose that  $X' \subset X$  and  $L'$  is the restriction of  $L$  to  $X'$ . Then  $(X, L')$  is an  $\mathcal{S}^*$ -space. Moreover, on  $X'$  the Fréchet topology induced by  $L'$  and the subspace topology of the Fréchet topology on  $X$  induced by  $L$  are the same.*

## Appendix B

# Complements of intervals in $\mathbb{R}^3$

In this appendix we prove that the complement of countably many intervals in  $\mathbb{R}^3$  is path connected and locally path connected. We apply this fact in Section 6.2 when studying the connectivity of  $\mathcal{PABL} \subset \mathcal{PML}(\Sigma_{0,5}) \cong \mathbb{S}^3$ .

An *interval* in  $\mathbb{R}^n$  is the image of a linear path  $p: I \rightarrow \mathbb{R}^n$ . Given a countable collection  $\{J_i : i \in \mathbb{N}\}$  of lines, intervals and points in  $\mathbb{R}^n$ , we define

$$\mathbb{J} = \mathbb{R}^n - \bigcup_{i \in \mathbb{N}} J_i.$$

**Lemma B.1.** *If  $n \geq 2$  then  $\mathbb{J}$  is dense in  $\mathbb{R}^n$ .*

*Proof.* Take any  $x \in \mathbb{R}^n$ . All but countably many lines through  $x$  intersect  $\mathbb{J}$  in a cocountable set. Suppose that  $\ell$  is one of these lines, then there exist  $x_i \in \mathbb{J} \cap \ell$  such that  $x_i \rightarrow x$ . This holds for every  $x \in \mathbb{R}^n$ , hence  $\mathbb{J}$  is dense in  $\mathbb{R}^n$ .  $\square$

From now on we assume that  $n = 3$ . A *convex polyhedron* is a 3-dimensional convex polytope, that is, it is the intersection of finitely many closed half-spaces in  $\mathbb{R}^3$  such that the interior is non-empty. Write  $\mathbb{J}_P$  for the intersection of  $\mathbb{J}$  with a convex polyhedron  $P$ .

**Lemma B.2.** *Suppose that  $x, y \in \mathbb{J}_P$  and  $z \in P$ . For any  $\varepsilon > 0$  there exist linear paths  $q_1, q_2: I \rightarrow \mathbb{R}^3$  with image contained in  $\mathbb{J}_P$  such that  $q_1(0) = x$ ,  $q_1(1) = q_2(0)$ ,  $q_2(1) = y$  and  $|z - q_1(1)| < \varepsilon$ .*

*Proof.* Fix  $x, y \in \mathbb{J}_P$ ,  $z \in P$  and  $\varepsilon > 0$ . Let  $A$  be a plane through  $z$  such that  $A_P = A \cap P$  is 2-dimensional and  $x, y \notin A_P$ . We will find a point  $a \in A_P$  at distance at most  $\varepsilon$  from  $z$  such that the concatenation of the linear paths connecting  $x$  to  $a$  and  $a$  to  $y$  is contained in  $\mathbb{J}$ .

For every  $i \in \mathbb{N}$  and  $v \in \{x, y\}$  there is a unique line or plane through  $v$  and  $J_i$ . Let  $\ell_{i,v}$  be the intersection of that line or plane with  $A$ . Lemma B.1 gives that the set

$$\mathbb{L} = A - \bigcup_{i \in \mathbb{N}} \ell_{i,x} \cup \ell_{i,y}$$

is dense in  $A$ . Choose  $a \in P \cap \mathbb{L}$  with  $|a - z| < \varepsilon$ . Let  $p_1$  be the linear path connecting  $x$  to  $a$  and let  $p_2$  be the linear path from  $a$  to  $y$ . These paths have the required properties.  $\square$

We use the lemma to show the following proposition.

**Proposition B.3.** *For any path  $p: I \rightarrow P$ ,  $\varepsilon > 0$  and  $x, y \in \mathbb{J}_P$  with  $d(x, p(0)) < \varepsilon$ ,  $d(y, p(1)) < \varepsilon$ , there exists a path*

$$q: I \rightarrow \mathbb{J}_P$$

such that  $q \subset N_\varepsilon(p)$ ,  $q(0) = x$  and  $q(1) = y$ .

*Proof.* Let  $p: I \rightarrow P \subset \mathbb{R}^3$  be a path and suppose that  $\varepsilon > 0$ . For  $n \in \mathbb{N}$  let  $p': I \rightarrow \mathbb{R}^3$  be the path that is the concatenation of  $n$  linear paths  $p'_1, p'_2, \dots, p'_n$ , where  $p'_i$  connects  $p(\frac{i-1}{n})$  and  $p(\frac{i}{n})$ . Choose  $n \in \mathbb{N}$  sufficiently large such that  $p' \subset N_{\frac{\varepsilon}{2}}(p)$ . Define  $x_0 = x$ ,  $x_n = y$  and for every  $1 \leq i < n$  pick  $x_i \in \mathbb{J}_P$  such that  $|x_i - p(\frac{i}{n})| < \frac{\varepsilon}{2}$ . By Lemma B.2, we can replace each interval  $p'_i$  by a path  $q_i$  that is a concatenation of two intervals in  $\mathbb{J}_P$ , connecting  $x_{i-1}$  via a point in  $N_{\frac{\varepsilon}{2}}(p(\frac{i}{n}))$  to  $x_i$ . This path  $q_i$  is contained in an  $\frac{\varepsilon}{2}$ -neighbourhood of  $p'_i$ . Define  $q: I \rightarrow \mathbb{J}_P$  as the concatenation  $q_1 * q_2 * \dots * q_n$ . Then  $q \subset N_{\frac{\varepsilon}{2}}(q') \subset N_\varepsilon(p)$ ,  $q(0) = x$  and  $q(1) = y$ .  $\square$

The proposition has the following consequence.

**Corollary B.4.** *The complement of countably many intervals and points in a convex polyhedron is path connected and locally path connected.*

*Proof.* Let  $P$  be a convex polyhedron and let  $\mathbb{J}_P$  be the complement in  $P$  of a countable collection of intervals and points. Path connectivity of  $\mathbb{J}_P$  is immediate from Lemma B.2. We proceed with local path connectivity. Suppose that  $x \in \mathbb{J}_P$  and let  $\delta > 0$ . Let  $y \in \mathbb{J}_P$  with  $|x - y| < \frac{1}{2}\delta$ . The interval between  $x$  and  $y$  is contained in  $P$ , so applying Proposition B.3 with  $\varepsilon = \frac{1}{2}\delta$  we find that  $x$  and  $y$  can be connected by a path in  $\mathbb{J}_P$  that stays  $\delta$ -close to  $y$ .  $\square$

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